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On Composition Operators on $N^+(\Omega)$

عن المؤثرات المركبة على $N^+(\Omega)$

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Abstract

Let $N(\Omega)$ denote the class of analytic functions f in a domain Ω, contained in the complex numbers C, such that $\log(1+|f|)$ has a harmonic majorant. The subclass $N^+(\Omega)$ of $N(\Omega)$ consists of all f such that $\log(1+|f|)$ has a quasi-bounded harmonic majorant. Let ϕ be a non-constant analytic function from Ω into itself. Define the composition operator C_ϕ on $N(\Omega)$ by $C_\phi f = f \circ \phi$, $\forall\ f \in N(\Omega)$. Then C_ϕ maps $N^+(\Omega)$ into itself. Here we characterize the invertibility of C_ϕ when Ω is finitely connected with boundary Γ consisting of disjoint analytic simple closed curves and we give a necessary condition for the density of the range of C_ϕ in $N^+(\Omega)$. Moreover, we consider linear isometries on $N^+(\Omega)$ and their relation to C_ϕ.

لتكن $N(\Omega)$ ترمز الى فئة الدوال التحليلية f في المجال المفتوح والمترابط Ω والمحتوى في الأعداد العقدية C وبحيث أن الدالة $\log(1+|f|)$ محدوده من أعلى بدالة توافقية. الفئة الجزئية $N^+(\Omega)$ من $N(\Omega)$ تتكون من جميع الدوال f حيث أن $\log(1+|f|)$ تكون محدوده من أعلى بدالة توافقية شبه محدوده. لتكن ϕ دالة غير ثابته وقابلة للإشتقاق ومن Ω إلى نفسها.إذا عرفنا المؤثر المركب C_ϕ على $N(\Omega)$ حسب القانون $C_\phi f = f \circ \phi$ لكل f منتمية الى $N(\Omega)$ فإنه ينقل $N^+(\Omega)$ الى نفسها. هنا سنصف انعكاس C_ϕ عندما تكون Ω محدودة الترابط وحدها Γ يتكون من منحنيات تحليلية مبسطه مغلقة ومنفصلة مثنى مثنى وسنقدم شرط لازم من أجل أن

يكون مدى C_ϕ كثيفا في $N^+(\Omega)$. بالاضافة الى ذلك سندرس التقايسات الخطية علـــى $N^+(\Omega)$ وعلاقتها بالمؤثر المركب C_ϕ .

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1. Introduction and Preliminaries

Let Ω be an open connected subset of the complex plane which we call a domaim. The Nevanlinna Calss $N(\Omega)$ consists of all functions f analytic in Ω such that log $(1+|f|)$ has a harmonic majorant. The subclass $N^+(\Omega)$ of $N(\Omega)$ consists of all f such that $\log(1+|f|)$ has a quasi-bounded harmonic majorant which means that $\log(1+|f|)$ is a pointwise increasing limit of a sequence of non-negative bounded harmonic functions.

Let u_f be the least harmonic majorant of $\log(1+|f|)$ where $f\in N(\Omega)$ and z_0 be a fixed point of Ω which we call the reference point. Define

$$\|f\| = u_f(z_0),\ \forall\ f\in N(\Omega).$$

Then $\|\ \ \|$ is a quasi-norm on $N(\Omega)$ and $N(\Omega)$ is a complete metric space which is an algebra over C. Furthermore, $N^+(\Omega)$ is an algebra over C and a topological vector space with complete translation invariant metric , i.e , an F-space (see[1]).

For p>0, $H^p(\Omega)$ consists of all functions f analytic in Ω such that $|f|^p$ has a harmonic majorant. Let w_f be the least harmonic majorant of $|f|^p$ where $f\in H^p(\Omega)$ and define

$$\|f\|_p=w_f(z_0)\ ,\ \forall f\in H^p\ 0<p<\infty.$$

Then $\|\ \ \|_p$ is a norm on $H^p(\Omega)$ which makes it a Banach space when $1\le p<\infty$. We have [1,p.259]

$$\bigcup_{p>0} H^p(\Omega)\subseteq N^+(\Omega)\subseteq N(\Omega)$$

If Ω is finitely connected with boundary Γ consisting of disjoint analytic simple closed curves and $R(\Omega)$ denotes the rational functions with poles off $\Omega\cup\Gamma$, then $R(\Omega)$ is dense in $N^+(\Omega)$ (see [2]). This implies that $H^p(\Omega)$, p>0 , and $H^{\infty}(\Omega)$, the space of all bounded analytic functions in Ω, are also dense in $N^{+}(\Omega)$.

Let ϕ be a non-constant analytic function from the domain Ω into itself. The composition operator C_ϕ on $N(\Omega)$ is defined by

$$C_\phi f = f o \phi, \ \forall f \in N(\Omega).$$

Its quasi-norm is defined by

$$\| C_\phi \| = \inf \{ M : \| f o \phi \| \leq M \| f \|, \ f \in N(\Omega) \}$$

We say that C_ϕ is bounded if $\| C_\phi \| < \infty$. In [3] (Theorem 4.1, p. 265) it is shown that C_ϕ is a bounded, hence continuous, linear operator on $N(\Omega)$ and maps $N^{+}(\Omega)$ into itself. Moreover, some results about compactness of C_ϕ are given there. We note that C_ϕ is always 1-1 on $N^{+}(\Omega)$ for if $C_\phi f = C_\phi g$, f, g $\in N^{+}(\Omega)$, then f and g agree on $\phi(\Omega)$ which is a non-empty open subset of Ω and hence they are equal on Ω. Moreover, C_ϕ maps $H^{p}(\Omega)$, $0 < p \leq \infty$, into itself.

For more information on the $H^{p}(\Omega)$ case one can see for example [1] and [5]. Here we characterize the invertibility of C_ϕ when Ω is finitely connected with boundary Γ consisting of disjoint analytic simple closed curves and we give a necessary condition for the density of the range of C_ϕ in $N^{+}(\Omega)$. Also, we consider linear isomtries on $N^{+}(\Omega)$and their relations to C_ϕ.

2. Invertibility of C_ϕ and density of its range

In this section we assume that Ω is finitely connected with boundary Γ consisting of disjoint analytic simple closed curves and ϕ is a non-constant analytic function from Ω into itself. We prove the following results.

Theorem 2.1: C_ϕ is invertible on $N^{+}(\Omega)$ if and only if ϕ is a conformal self-equivalence of Ω, in which case $C^{-1}_\phi = C_\psi$ where $\psi = \phi^{-1}$.

Proof: Let ϕ be a conformal self-equivalence of Ω and $\psi = \phi^{-1}$. Then $(C_\psi o C_\phi)(f) = (C_\phi o C_\psi)(f) = f$, $\forall f \in N^{+}(\Omega)$, i. e., C_ϕ is invertible and $C^{-1}_\phi = C_\psi$.

Conversely, suppose C_ϕ is invertible on $N^+(\Omega)$ with inverse $C^{-1}_\phi=S$. Let $f\in H^\infty(\Omega)$ and $Sg=f$. Then $C_\phi f=g\in H^\infty(\Omega)$ which means that S is the inverse of C_ϕ when both S and C_ϕ are restricted to $H^\infty(\Omega)$. Since S restricted to $H^\infty(\Omega)$ is the inverse of an algebra automorphism of $H^\infty(\Omega)$, S itself must be an algebra automorphism of $H^\infty(\Omega)$ (see [5] (p.52)). Then by [4] (Theorem 9, p.335) there exists a conformal self-equivalence ψ of Ω such that

$$Sf=fo\psi=C_\psi f,\ \forall\ f\in H^\infty(\Omega).$$

The continuity of C_ψ on $N^+(\Omega)$ (see [3] (Theorem 4.1, p.265)) and the density of $H^\infty(\Omega)$ in $N^+(\Omega)$ imply that $S=C_\psi$ on $N^+(\Omega)$. Finally noting that the function $f(z)=z$ is in $N^+(\Omega)$ it follows that $\phi^{-1}=\psi$.

Theorem 2.2: If C_ϕ: $N^+(\Omega)\to N^+(\Omega)$ has dense range, then ϕ is 1-1.

Proof: Let z_1,z_2 be two distinct points of Ω. We show that there exists $f\in H^\infty(\Omega)$ such that $f(z_1)\neq f(z_2)$. Since Ω supports non-constant analytic function there exists a non-constant analytic function $f_1\in H^\infty(\Omega)$. Let $f_2(z)=f_1(z)-f_1(z_1)$. If $f_2(z_2)\neq 0$, then there is nothing to prove. So assume $f_2(z_2)=0$. One can choose a natural number n such that $f(z)=(z-z_1)^{-n}f_2(z)$ is in $H^\infty(\Omega)$ with $f(z_1)\neq 0$ and $f(z_2)=0$.

Since range C_ϕ is dense in $N^+(\Omega)$ there exists a sequence $\{g_n\}$ in $N^+(\Omega)$ such that $\{C_\phi g_n\}$ converges to f in $N^+(\Omega)$ and hence by [1] (Corollary 2.4, p.261) it converges to f uniformly on compact subsets of Ω. Since $\{z_1,z_2\}$ is a compact subset of Ω, letting $\varepsilon=|f(z_1)-f(z_2)|>0$ implies that there exists a natural number k such that

$$n\geq k\Rightarrow|g_n(\phi(z_i))-f(z_i)|<\varepsilon/2,\ i=1,2.$$

Therefore, $|g_n(\phi(z_1))-g_n(\phi(z_2))|>0,\ \forall\ n\geq k$. Thus $\phi(z_1)\neq\phi(z_2)$ which shows that ϕ is 1-1.

3. Linear Isometries on $N^+(\Omega)$

Let Ω be a domain in the complex plane for which $H^p(\Omega)$ is nontrivial and ϕ is a nonconstant analytic mapping of Ω into itself. We start by listing some results about linear isometries of $H^p(\Omega)$, $p>0$. A linear isometry A of $H^p(\Omega)$ into $H^p(\Omega)$ is a linear operator satisfying

$$\|Af\|_p=\|f\|_p \ , \ \forall\ f\in H^p(\Omega)$$

The composition operator C_ϕ is a linear isometry of $H^p(\Omega)$ onto $H^p(\Omega)$ if and only if ϕ is both 1-1 , onto , and $\phi(z_0)=z_0$ (see [1] (p. 228)). Let Γ be the boundary of Ω. If ω and ω' are the harmonic measures on Γ with respect to z_0 and $\phi^{-1}(z_0)$ respectively , then by Theorem 1.3 (p.211) and Corollary 1.4 (p. 212) of [1] we have the following results.

Theorem 3.1: Let $0<p<\infty$, $p\neq 2$, and let A be a linear isometry of $H^p(\Omega)$ into $H^p(\Omega)$. Then there is an analytic function ϕ mapping Ω into Ω and a function $F\in H^p(\Omega)$ with

$$Af=FC_\phi f, \tag{3.1}$$

for each $f\in H^p(\Omega)$. Moreover, ϕ maps Γ into Γ a. e. ω and ϕ and F are related by

$$\omega(E) = \int_{\phi^{-1}(E)} |F|^p \, d\omega ,$$

for each measurable set E in Γ .

Corollary 3.2: Let Ω be bounded by a finite number of disjoint analytic simple closed curves. Let A be a linear isometry of $H^p(\Omega)$ onto $H^p(\Omega)$, $1\leq p<\infty$, $p\neq 2$. Then $\forall\ f\in H^p(\Omega)$

$$Af=\lambda FC_\phi f, \tag{3.2}$$

where λ is a complex scalar , and ϕ is a 1-1 analytic mapping of Ω onto Ω and F is an outer function in $H^p(\Omega)$ with

$$|F|^p = \frac{d\omega'}{d\omega}. \tag{3.3}$$

In particular , if $A1=1$, then $\forall\ f\in H^p(\Omega)$

$$Af = \lambda C_\phi f. \tag{3.4}$$

A linear isometry A of $N^+(\Omega)$ into $N^+(\Omega)$ is a linear operator satisfying

$$\|Af\| = \|f\| \quad , \forall \; f \in N^+(\Omega).$$

Now we prove the following results.

Theorem 3.3: Let Ω be bounded by a finite number of disjoint analytic simple closed curves namely Γ. If A is a linear isometry of $N^+(\Omega)$ into $N^+(\Omega)$, then the restriction of A to $H^1(\Omega)$ is a linear isometry of $H^1(\Omega)$ into $H^1(\Omega)$.

Proof: Let $f \in H^1(\Omega)$ be fixed and $Af = g \in N^+(\Omega)$. By proposition 3.3 of [3] for $f \in N^+(\Omega)$, we have

$$\|f\| = \int_\Gamma \log(1+|f^*|)d\omega \,,$$

where f^* is the boundary values function of f.

Since A is a linear isometry of $N^+(\Omega)$, for n =1,2,3 ...., we obtain

$$\int_\Gamma \log(1+\left|\frac{f^*}{n}\right|) \; d\omega = \left\|\frac{f}{n}\right\| = \left\|\frac{g}{n}\right\| = \int_\Gamma \log(1+\left|\frac{g^*}{n}\right|)d\omega \,.$$

Multiplying by n it follows that

$$\int_\Gamma \log(1+\left|\frac{f^*}{n}\right|)^n d\omega = \int_\Gamma \log(1+\left|\frac{g^*}{n}\right|)^n d\omega \,.$$

The fact that $(1+\frac{x}{n})^n$ increases monotonically to e^x as $n \to \infty$ for any $x \geq 0$, the Monotone Convergence Theorem and Corollary 4.5 of [1] (p. 90), namely

$$\|f\|_1 = \int_\Gamma |f^*| d\omega, \; \forall \; f \in H^1(\Omega)$$

imply that

$$\|f\|_1 = \int_\Gamma |f^*| d\omega = \lim_{n\to\infty} \int_\Gamma \log\left(1 + \left|\frac{f^*}{n}\right|\right)^n d\omega$$

$$= \lim_{n\to\infty} \int_\Gamma \log\left(1 + \left|\frac{g^*}{n}\right|\right)^n d\omega$$

$$= \int_\Gamma |g^*| \, d\omega = \|g\|_1 = \|Af\|_1$$

Thus the restriction of A to $H^1(\Omega)$ is a linebv/ar isometry of $H^1(\Omega)$ into $H^1(\Omega)$.

As a consequence of Theorem 3.1 , Corollary 3.2 and Theorem 3.3 we obtain the following corollaries.

Corollary 3.4: Let Ω be bounded by a finite number of disjoint analytic simple closed curves. Let A be a linear isometry of $N^+(\Omega)$ into $N^+(\Omega)$. Then

1. There is an analytic function ϕ mapping Ω into Ω and a function $F \in H^1(\Omega)$ such that (3.1) holds, $\forall\ f \in N^+(\Omega)$.
2. If A is onto , then there is a complex scalar λ, a 1-1 analytic mapping ϕ of Ω onto Ω , and an outer fuction F in $H^1(\Omega)$ such that (3.3) holds and (3.2) holds $\forall$, $f \in N^+(\Omega)$.

   In particular , if A1=1 , then (3.4) holds, $\forall\ f \in N^+(\Omega)$.

Proof: Let A be a linear isometry of $N^+(\Omega)$ into $N^+(\Omega)$. Then by Theorem 3.3, A is a linear isometry of $H^1(\Omega)$ into $H^1(\Omega)$. Thus by Theorem 3.1, there is an analytic function ϕ mapping Ω into Ω and a function $F \in H^1(\Omega)$ such that (3.1) holds, $\forall\ f \in H^1(\Omega) \subseteq N^+(\Omega)$. We show that (3.1) holds, $\forall\ f \in N^+(\Omega)$ and the rest is clear.

Let $g \in N^+(\Omega)$ and define a multiplication operator M_g on $N^+(\Omega)$ by $M_g f = gf$, $\forall\ f \in N^+(\Omega)$. Since $N^+(\Omega)$ is an algebra the closed graph theorem implies that M_g is a continuous linear operator from $N^+(\Omega)$ into $N^+(\Omega)$. Therefore ,

$M_F o C_\phi$ is a continuous linear operator from $N^+(\Omega)$ into $N^+(\Omega)$ since C_ϕ is a continuous linear operator from $N^+(\Omega)$ into $N^+(\Omega)$.

Since $H^1(\Omega)$ is dense in $N^+(\Omega)$ and (3.1) holds, $\forall f \in H^1(\Omega)$, it follows that (3.1) holds $\forall f \in N^+(\Omega)$.

Corollary 3.5: Let Ω be bounded by a finite number of disjoint analytic simple closed curves. Then C_ϕ is an isometry of $N^+(\Omega)$ onto $N^+(\Omega)$ iff ϕ is both 1-1, onto , and $\phi(z_0)=z_0$.

Proof: By [1] (p.228), we have, C_ϕ is an isometry of $H^p(\Omega)$, $1 \le p < \infty$, onto $H^p(\Omega)$ iff ϕ is both 1-1, onto , and $\phi(z_0)=z_0$.

Suppose C_ϕ is an isometry of $N^+(\Omega)$ onto $N^+(\Omega)$. Then by Theorem 3.3, C_ϕ is an isometry of $H^1(\Omega)$ onto $H^1(\Omega)$. Hence , ϕ is both 1-1 , onto and $\phi(z_0)=z_0$.

For the converse just note that, $\forall f \in N^+(\Omega)$

$$||f||=u_f(z_0)= u_f(\phi(z_0))=(u_f o\phi)(z_0)=u_{fo\phi}(z_0)=||fo\phi||=||C_\phi f||.$$

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