On Multipliers of Orlicz Spaces

حول مضاعفات فضاءات أورلكس

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Abstract

Let (T, M, μ) be a finite positive measure space, *X* a Banach space, ϕ a modulus function and $f : T \to X$ a strongly measurable function. The Orlicz space is $L^{\phi}(\mu, X) = \left\{ f : \int_T \phi(\|f(t)\|) d\mu(t) < \infty \right\}$.The space of Bochner *p*-integrable functions, $1 \le p < \infty$ is $L^p(\mu, X) = \left\{ f : \int_T \|f(t)\|^p d\mu(t) < \infty \right\}$.Also, $L^{\infty}(\mu, X) = \left\{ f : ess \sup_{t \in T} \|f(t)\| < \infty \right\}$.

When X is a Banach algebra we show that the multipliers $M(L^{\phi}(\mu, X))$ of $L^{\phi}(\mu, X)$ is $L^{\infty}(\mu, X)$ if $\phi(a)\phi(b) \le \phi(ab)$ for all a≥ 1 and b$\ge$ 0 .Also, $M(L^{\phi}(\mu, X)) = L^{\phi}(\mu, X)$ if $\phi(ab) \le \phi(a) + \phi(b)$ for all a,b in $[0, \infty)$ which generalizes the special case X being the complex numbers **C** .When (T, M, μ) is also non-atomic we show that $f^2 \in L^{\phi}(\mu, X)$ for all $f \in L^{\phi}(\mu, X)$ iff $\limsup_{x \to \infty} \frac{\phi(x^2)}{\phi(x)} < \infty$.

Moreover, $M(L^{\phi}(\mu, X)) = L^{\phi}(\mu, X)$ iff $\lim_{x \to \infty} \sup \frac{\phi(x^2)}{\phi(x)} < \infty$ when X is commutative. This generalizes the special case T= [0,1] and X=**C.**

ملخص

ليكن (T,M, μ) فضاءا قياسيا موجبا ومنتهيا و X فضاء بناخ و ϕ اقتران مطلق القيمة و $f : T \to X$ اقتران قياسي بقوة.فان فضاء أورلكس هو $L^{\phi}(\mu, X) = \left\{ f : \int_T \phi(\|f(t)\|) d\mu(t) < \infty \right\}$.

وفضاء بوخنر عندما $1 \le p < \infty$ هو $L^p(\mu, X) = \left\{ f : \int_T \|f(t)\|^p d\mu(t) < \infty \right\}$. أيضا $L^\infty(\mu, X) = \left\{ f : ess \sup_{t \in T} \|f(t)\| < \infty \right\}$. اذا كان X فضاء بناخ الجبري فسنثبت،ان شاء اللة، أن المضاعفات $M(L^\phi(\mu, X))$ ل $L^\phi(\mu, X)$ هي $L^\infty(\mu, X)$ عندما $\phi(a)\phi(b) \le \phi(ab)$ لكل $a \ge 1$ و $b \ge 0$. كذلك $M(L^\phi(\mu, X)) = L^\phi(\mu, X)$ عندما $\phi(ab) \le \phi(a) + \phi(b)$ لكل a و b في $[0, \infty)$ مما يعتبر تعميما للحالة الخاصة التي تكون فيها X هي الاعداد العقدية **C**. عندما يكون (T, M, μ) فضاءا غير نووي أيضا سنثبت ،ان شاء اللة ،أن $f^2 \in L^\phi(\mu, X)$ لكل $f \in L^\phi(\mu, X)$ اذا وفقط اذا $\lim_{x \to \infty} \sup \frac{\phi(x^2)}{\phi(x)} < \infty$ وكذلك $L^\phi(\mu, X) = M(L^\phi(\mu, X))$ اذا وفقط اذا $\lim_{x \to \infty} \sup \frac{\phi(x^2)}{\phi(x)} < \infty$ عندما يكون X تبديليا وهذا يعتبر تعميما للحالة الخاصة **X=C** و T=[0,1] .

1. Introduction

If ϕ is a strictly increasing continuous subadditive function on $[0, \infty)$ and satisfies $\phi(x) = 0$ if $x = 0$, then we call ϕ a modulus function. Let (T, M, μ) be a finite positive measure space, i.e., T is a set, M is a σ – algebra and μ is a positive measure with $\mu(T) < \infty$. If *X* is a Banach space, then a function $s : T \to X$ is called a simple function if its range contains finitely many distinct points $x_1, x_2, ..., x_n$ and $E_i = s^{-1}(\{x_i\})$, $i = 1,2,...,n$ are measurable sets. Such a function *s* can be written as $s = \sum_{i=1}^{n} x_i \chi_{E_i}$, where χ_{E_i} is the characteristic function of the set E_i and $E_i \cap E_j = \Phi$, for $i \ne j$, $i, j = 1,2,...,n$.

A function $f : T \to X$ is said to be strongly measurable if there exists a sequence $\{s_n\}$ of simple functions such that $\lim_{n \to \infty} \|s_n(t) - f(t)\| = 0$ a.e.

The Orlicz space $L^{\phi}(\mu, X)$ is the set of all strongly measurable functions *f* with

$$\|f\|_{\phi} = \int_T \phi(\|f(t)\|) d\mu(t) < \infty .$$

If for all $f, g \in L^{\phi}(\mu, X)$ we define $d(f,g) = \|f - g\|_{\phi}$, then d is a metric on $L^{\phi}(\mu, X)$ under which it becomes a complete topological vector space [1,p.70]. For $1 \le p < \infty$, $L^{p}(\mu, X)$ will denote the Banach space of (equivalence classes of) strongly measurable functions *f* such that $\int_T \|f(t)\|^p d\mu(t) < \infty$. The norm in $L^{p}(\mu, X)$ is given by

$$\|f\|_{p} = \left(\int_T \|f(t)\|^p d\mu(t) \right)^{\frac{1}{p}} .$$

The essentially bounded strongly measurable functions *f* form the Banach space $L^{\infty}(\mu, X)$ with norm given by $\|f\|_{\infty} = \underset{t \in T}{ess \sup} \|f(t)\|$.

If ϕ is the modulus function $\phi(x) = x^p$, $0 < p \le 1$, then $L^{\phi}(\mu, X)$ is the space $L^{p}(\mu, X)$. Since [2, p.159], for any modulus function ϕ, $\limsup\limits_{x \to \infty} \frac{\phi(x)}{x} \le \phi(1)$,

it follows that $L^{1}(\mu, X) \subseteq L^{\phi}(\mu, X)$. When X is a Banach algebra (see [5]) a multiplier of $L^{\phi}(\mu, X)$ is a strongly measurable function g: T$\to$X such that $gf \in L^{\phi}(\mu, X)$ for all $f \in L^{\phi}(\mu, X)$. We denote the set of all multipliers of $L^{\phi}(\mu, X)$ by $M(L^{\phi}(\mu, X))$. For X = **C**, the complex numbers, $M(L^{\phi}(\mu, C) = M(L^{\phi})$ were studied in [2]. In this paper we show that some of the results in [2] still hold for $M(L^{\phi}(\mu, X))$. A measurable set A is called an atom if each of its measurable subsets has measure either 0 or $\mu(A)$. The measure space (T, M, μ) is called non-atomic if it contains no atoms. In [4,p.122] it is shown that if (T, M, μ) is a finite non-atomic measure space and 0< $\theta < \mu(T)$, then there exists a measurable set E such that $\mu(E) = \theta$. Using this we show that

$$\limsup_{x \to \infty} \frac{\phi(x^2)}{\phi(x)} < \infty \text{ iff } f^2 \in L^\phi(\mu, X) \text{ for all } f \in L^\phi(\mu, X)$$

when (T, M, μ) is a finite positive non-atomic measure space. This generalizes the main result in [3] where T=[0,1] and X=**C**.

2. Multipliers of $L^\phi(\mu, X)$

Lemma 2.1 Let X be Banach algebra .If g$\in M(L^1(\mu, X))$, then g$\in L^\infty(\mu, X)$.

Proof: Suppose that g $\in M(L^1(\mu, X))$. Let $f : T \to X$ be given by $f(t) = e \text{ for all } t \in T$, where e is the unit element of X. Then $f \in L^1(\mu, X)$ and

$$\int_T \| g(t) \| \, d\mu(t) = \int_T \| g(t) f(t) \| \, d\mu(t) = \| gf \|_1 < \infty .$$

Hence, g$\in$ $L^\infty(\mu, X)$.

The next results are generalizations of those in [2]. Without loss of generality we can assume that ϕ is an unbounded modulus function and ϕ (1)=1. For if ϕ is bounded, then $L^\phi(\mu, X)$ is the strongly measurable functions. Also, if $\phi(1) \neq 1$, then we can replace ϕ by $\frac{\phi}{\phi(1)}$.

Theorem 2.2 If $\phi(a)\phi(b) \le \phi(ab)$ for all $a \ge 1$ and $b \ge 0$, then

$M(L^{\phi}(\mu,X)) = L^{\infty}(\mu,X)$**,** where X is a Banach algebra.

Proof: Let g$\in L^{\infty}(\mu,X)$ and $f \in L^{\phi}(\mu,X)$ .Choose a natural number n such that $\|g\|_{\infty} < n$. Then since X is a Banach algebra we have

$$\|gf\|_{\phi} = \int_T \phi(\|g(t)f(t)\|)d\mu(t) \le \int_T \phi(\|g(t)\| \; \|f(t)\|)d\mu(t)$$

$$\le \int_T \phi(n\|f(t)\|)d\mu(t)$$

$$\le n\|f\|_{\phi} < \infty$$

Thus $L^{\infty}(\mu,X) \subseteq M(L^{\phi}(\mu,X))$. We note that $M(L^{\phi}(\mu,X)) \subseteq L^{\phi}(\mu,X)$ since if e is the unit element of X and $f(t) = e$ *for all* $t \in T$,then $f \in L^{\phi}(\mu,X)$ and gf = g$\in L^{\phi}(\mu,X)$ for all g$\in M(L^{\phi}(\mu,X))$.

Next, for g$\in M(L^{\phi}(\mu,X))$ and f$\in L^{1}(\mu,X)$ let $\tilde{g}(t) = \phi(\|g(t)\|)e$ and $h(t) = \phi^{-1}(\|f(t)\|)$ for all t$\in$T . Then

$$\|h\|_{\phi} = \int_T \|f(t)\| d\mu(t) = \| \; f \; \|_1 < \infty \quad .$$

Thus, $h \in L^{\phi}(\mu,X)$ and hence $gh \in L^{\phi}(\mu,X)$. If A={t:‖g(t)‖ >1}, then

$$\|\tilde{g}f\|_1 = \int_T \|\phi(\|g(t)\|)f(t)\| d\mu(t)$$

$$= \int_T \phi(\|g(t)\|)\|f(t)\| d\mu(t) = \int_T \phi(\|g(t)\|)\phi(\|h(t)\|)d\mu(t)$$

$$\le \int_A \phi(\|g(t)\|)\phi(\|h(t)\|)d\mu(t) + \int_{T\setminus A} \phi(\|g(t)\|)\phi(\|h(t)\|)d\mu(t)$$

$$\le \int_A \phi(\|g(t)\|\,\|h(t)\|)d\mu(t) + \int_{T\setminus A} \phi(\|h(t)\|)d\mu(t)$$

$$\le \|gh\|_{\phi} + \|h\|_{\phi} < \infty .$$

This shows that $\tilde{g} = \phi(\| g \|)e$ is a multiplier of $L^1(\mu, X)$. Thus $\tilde{g} \in L^\infty(\mu, X)$ by lemma 2.1. This implies that $g \in L^\infty(\mu, X)$ and $M(L^\phi(\mu, X)) \subseteq L^\infty(\mu, X)$. Therefore, $M(L^\phi(\mu, X)) = L^\infty(\mu, X)$.

Theorem 2.3 If $\phi(ab) \le \phi(a) + \phi(b)$ for all a, b$\in[0,\infty)$, then $M(L^\phi(\mu, X)) = L^\phi(\mu, X)$ where X is a Banach algebra.

Proof: As in theorem 2.1 we have $M(L^\phi(\mu, X)) \subseteq L^\phi(\mu, X)$. Let $g \in L^\phi(\mu, X)$. Then, for all $f \in L^\phi(\mu, X)$

$$\| gf \|_\phi = \int_T \phi(\| g(t) f(t) \|) d\mu(t) \le \int_T \phi(\| g(t) \| \, \| f(t) \|) d\mu(t)$$

$$\le \int_T (\phi(\| g(t) \|) + \phi(\| f(t) \|)) d\mu(t)$$

$$= \| g \|_\phi + \| f \|_\phi < \infty$$

Therefore, $g \in M(L^\phi(\mu, X))$. Thus $L^\phi(\mu, X) = M(L^\phi(\mu, X))$.

The following is a generalization of the main result in [3] from the Lebesgue measure on [0,1]=T and the complex numbers **C** to a non-atomic, finite, and positive measure space.

Theorem 2.4 Let (T, M, μ) be a finite positive, non-atomic measure space, and ϕ be a modulus function. Then

$$\limsup_{x\to\infty} \frac{\phi(x^2)}{\phi(x)} < \infty \text{ iff } f^2 \in L^\phi(\mu, X) \text{ for all } f \in L^\phi(\mu, X)$$

where X is a Banach algebra .

Proof: Suppose $\limsup_{x\to\infty} \frac{\phi(x^2)}{\phi(x)} = \infty$. Then there exists an increasing sequence $\{x_n\}$ such that $x_1 > 1$ and $\frac{\phi(x_n{}^2)}{\phi(x_n)} > n$ for all n = 1,2,3, …. Since

(T, M, μ) is a finite non-atomic measure space by lemma 10.12 [4,p.122] there exists a sequence $\{E_n\}$ of pairwise disjoint measurable sets such that

$$T = \cup_{n=1}^{\infty} E_n \text{ and } \mu(E_n) = \frac{\mu(T)}{n^2 \phi(x_n)} \quad \text{for all n=1,2,3,\ldots}$$

Define $f : T \to X$ by $f(t) = \sum_{i=1}^{\infty} x_i \chi_{E_i}(t)\, e$ for all $t \in T$ where e is the unit element of X . Then

$$\|f\|_{\phi} = \int_T \phi(\|f(t)\|) d\mu(t) = \sum_{n=1}^{\infty} \int_{E_n} \phi(\| \textstyle\sum_{i=1}^{\infty} x_i \chi_{E_i}(t)\, e \|) d\mu(t)$$

$$= \sum_{n=1}^{\infty} \int_{E_n} \phi(x_n) d\mu(t) = \sum_{n=1}^{\infty} \phi(x_n)\mu(E_n) = \sum_{n=1}^{\infty} \frac{\mu(T)}{n^2} < \infty$$

Therefore, $f \in L^{\phi}(\mu, X)$**.** Moreover,

$$\|f^2\|_{\phi} = \int_T \phi(\|f^2(t)\|) d\mu(t) = \sum_{n=1}^{\infty} \int_{E_n} \phi(\| (\textstyle\sum_{i=1}^{\infty} x_i \chi_{E_i}(t)\, e)^2 \|) d\mu(t)$$

$$= \sum_{n=1}^{\infty} \int_{E_n} \phi(x_n^2) d\mu(t)$$

$$= \sum_{n=1}^{\infty} \phi(x_n^2)\mu(E_n) \geq \sum_{n=1}^{\infty} n\phi(x_n)\mu(E_n) = \sum_{n=1}^{\infty} \frac{\mu(T)}{n} = \infty .$$

Thus, $f^2 \notin L^{\phi}(\mu, X)$**.** Therefore, if $f^2 \in L^{\phi}(\mu, X)$ for all $f \in L^{\phi}(\mu, X)$**,** then

$$\limsup_{x \to \infty} \frac{\phi(x^2)}{\phi(x)} < \infty .$$

Conversely, suppose $\limsup_{x \to \infty} \frac{\phi(x^2)}{\phi(x)} < \infty$. Then there exist constants M and K such that

$$\frac{\phi(x^2)}{\phi(x)} < M \text{ for all } x \geq K .$$

Let $f \in L^{\phi}(\mu, X)$ and A={t∈T: ‖f(t)‖≤K}.Then since X is a Banach algebra

$$\begin{aligned} \left\|f^2\right\|_{\phi} &= \int_T \phi(\left\|f^2(t)\right\|)d\mu(t) \leq \int_T \phi(\left\|f(t)\right\|^2)d\mu(t) \\ &= \int_A \phi(\left\|f(t)\right\|^2)d\mu(t) + \int_{T-A} \phi(\left\|f(t)\right\|^2)d\mu(t) \\ &\leq \phi(K^2)\mu(T) + \int_{T-A} M\phi(\left\|f(t)\right\|)d\mu(t) \\ &\leq \phi(K^2)\mu(T) + \mathrm{M} \, \| f \|_{\phi} < \infty . \end{aligned}$$

Therefore, $f^2 \in L^{\phi}(\mu, X)$ for all $f \in L^{\phi}(\mu, X)$.

Corollary 2.5 Let (T, M, μ) be a non-atomic, finite, positive measure space, X be a commutative Banach algebra, and ϕ be a modulus function. Then

$$M(L^{\phi}(\mu, X)) = L^{\phi}(\mu, X) \text{ iff } \limsup_{x\to\infty} \frac{\phi(x^2)}{\phi(x)} < \infty$$

Proof: Note that $L^{\phi}(\mu, X)$ is an algebra iff $f^2 \in L^{\phi}(\mu, X)$ for all

$f \in L^{\phi}(\mu, X)$ follows from $fg = \frac{1}{2}((f+g)^2 - f^2 - g^2)$ and the corollary follows from theorem 2.4.

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