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On Lipschitz Functions of subnormal Operators

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الملخص

اقترانات ليبشتز للمؤثرات الطبيعية الجزئية

أثبتنا في هذا البحث النظرية التالية:

إذا كان S مؤثراً من النوع الطبيعي الجزئي يعمل على فضاء هلبرت H وكانت f اقتراناً تحقق شرط ليبشتز على جوار مفتوح يحتوي المجموعة الطيفية للمؤثر S فإن:

$$\|f(S)X - Xf(S)\|_2 \le k\|SX - XS\|_2$$

حيث $0<k$ ، ولكل مؤثر X يعمل على H حيث $\|\|_2$ هو مقياس هلبرت-شميدت.

كذلك فإن البحث يحتوي على بعض النتائج والنظريات ذات العلاقة بالنظرية الرئيسية.

ABSTRACT

We prove the following:

Theorem: Let S be a subnormal operator on an infinite dimensional separable complex Hilbert space H. Let f be analytic and Lipschitz on a neighborhood E of the spectrum $\sigma(S)$ of S. Then

$$\|f(S)X - Xf(S)\|_2 \le k\|SX - XS\|_2,$$

for some $k > 0$ and for every $X \in B(H)$, the algebra of all bounded linear operators on H, where $\|\|_2$ is the Hilbert-Schmidt norm.

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INTRODUCTION

Let H denote an infinite dimensional complex Hilbert space, B(H) denote the algebra of all bounded linear operators on H. For a compact subset E of the complex plane C, let R(E) denote the algebra of all bounded analytic functions defined on E; and L(E) the set of all functions which satisfies Lipschitz condition on E, i,e., $f \in L(E)$ if and only if there is a real number $k > 0$ such that:

$$|f(z)-f(t)\leq k|z-t|,$$

for all $z,t \in E$.

In [1], it is proved that if A is a self adjoint operator and $f \in L(\sigma(A))$ then

$$\|f(A)X - Xf(A)\|_2 \leq k\|AX - XA\|_2$$

for some $k>0$, and every $X \in C_2$, the class of Hilbert-Schmidt operators, where $\sigma(A)$ is the spectrum of A and $\|.\|_2$ is the Hilbert-Schmidt norm, defined by

$$\|T\|_2 = \left(\sum_{n=1}^{\infty} \|Te_n\|^2\right)^{\frac{1}{2}},$$

where (e_n) is an orthonormal basis of H, and T is any operator on H.

In [6], this result has been generalized to normal operators as follows;

Theorem: If N is a normal operator on H, and $f \in L(\sigma(N))$ then,

$$\|f(N)X - Xf(N)\|_2 \leq k\|NX - XN\|_2 ,$$

for some $k > 0$, and every $X \in B(H)$, where an operator N is said to be normal if $NN^* = N^*N$, (The operator N^* is the adjoint of N).

For the proof see [6].

In this article we generalize the above theorem to subnormal operators on H. An operator S on H is called subnormal if there is a normal extension N on a Hilbert space $K \supset H$. If N is a minimal normal extension of S, i.e., if N has no reducing subspace L of K which contains H properly; then N has the following representation

$$N = \begin{bmatrix} S & A \\ 0 & T^* \end{bmatrix}, \qquad (1)$$

on $K = H \oplus H^{\perp}$, where A is some operator from $H^{\perp}$ to H. In this case T is called the dual of S, (and T^* is the adjoint of T).

A subset E of the complex plane is called a spectral set of an operator $S \in B(H)$ if and only if, the spectrum $\sigma(S)$ of S is a subset of E and

$$\|f(S)\| \le \|f\|_\infty = \sup\{|f(z)| : z \in E\},$$

for every rational function f with poles off E, or equivalently $f \in R(E)$.

Conway [4] proved that if S is a subnormal operator with dual T and E is a spectral set of S then,

$$f(N) = \begin{bmatrix} f(S) & A' \\ 0 & f(T^*) \end{bmatrix}, \tag{2}$$

for any $f \in R(E)$, where A' is some operator from $H^\perp$ to H.

The following theorem is the main result of this article.

Theorem 1: Le S be a subnormal operator on H. If $f \in R(E) \cap L(E)$, then

$$\|f(S)X - Xf(S)\|_2 \le k\|SX - XS\|_2 \tag{3}$$

for some $k>0$, and every $X \in B(H)$.

Proof: Since S is a subnormal operator on H, therefore there is a sequence of normal operators (N_i) such that (N_i) converges to S, in the strong operator topology, denoted by sot; (see Conway [3] pages 123, 124, 125).

We claim that if $f \in R(E)$, then $f(N_i)$ converges to f(S) (sot). Indeed; by the construction of (N_i),

$$N_i = v_i^{-1} N v_i, \quad I=1,2,3,\ldots, \tag{4}$$

where N is a normal extension of S to a Hilbert space $K \supset H$, (See the proof of Theorem 1.17, page 125 in Conway [3]), and

v_i: $H \rightarrow K$,

is an isomorphism such that v_i=identity on H_i, where

$H_i = V\{e_1,\ldots,e_i\ Se_1,\ldots Se_i\} = Span\{e_i,\ldots,e_i,Se_1,\ldots,Se_i\}$.

This means that N_i and N are similar (where two operators S,T are said to be similar if there is an invertible operator $X: S=X^{-1}TX$). Thus, by problem 75 in Halmos [5], (page 42, which says that two similar operators has the same spectrum).

$$\sigma(N_i) = \sigma(N), \quad i=1,2,3,\ldots$$

Since N is a normal extension of S, it follows that $\sigma(N) \subset \sigma(S)$. Therefore

$\sigma(N_i) \subset E$, i=1,2,3,...

Thus $f(N_i)$ is well-defined for every f in R(E) and every i=1,2,3,...

Now, since $f \in R(E)$, there is a sequence of polynomials (P_n) such that (P_n) converges to f, uniformly on compact subsets of E. This implies, using Conway [2] (page 206, by Riesz functional calculus theorem) that $P_n(N_i)$ converges to $f(N_i)$, in the uniform operator topology (denoted by (uot).), i.e., given i, we have:

$$\|P_n(N_i) - f(N_i)\| \to 0, \; n \to \infty, i = 1,2,3,\ldots$$
$$\|P_n(S_i) - f(S)\| \to 0, \; n \to \infty.$$

Thus, for every $\epsilon > 0$, and nonzero $e \in H$, there is J such that:

$$\|P_n(N_i) - f(N_i)\| < \frac{\epsilon}{3\|e\|}, \; n > J, \; i = 1,2,3,\ldots$$

$$\|P_n(S_i) - f(S)\| < \frac{\epsilon}{3\|e\|}, \; n > J.$$

For arbitrary $e \in H$, and $n_0 = J+1$, this implies that:

$$\begin{aligned} \|(f(N_i) - f(S))e\| &\le \|(f(N_i) - P_{n_0}(N_i))e\| + \|(P_{n_0}(N_i) - P_{n_0}(S))e\| \\ &\quad + \|(P_{n_0}(S) - f(S))e\| \qquad (5) \\ &< 2\epsilon/3 + \|(P_{n_0}(N_i) - P_{n_0}(S))e\|. \end{aligned}$$

Since P_{n_o} is a polynomial and since the strong convergence is sequentially continuous (see Halmos [5], problem, 113 page 62), one concludes that

$$P_{n_o}(N_i) \to P_{n_o}(S), \text{ as } i \to \infty, \text{ (sot)},$$

and thus to every $\epsilon>0$, $\exists$ J such that:

$$\left\| \left(P_{n_o}(N_i) - P_{n_o}(S)\right) e \right\| < \epsilon/3, \text{ for } i > J. \qquad (6)$$

Combining (5) and (6) gives the convergence of $f(N_i)$ to $f(S)$, (sot), for every $f \in R(E)$, (notice that this result is trivial for e=0).

Now, applying problem 112 page 62 in Halmos [5], (which inserts that product is left and right continuous with one argument fixed, $A \to AB$ or $A \to BA$) one obtains the following:

$$N_iX \to SX, \text{ sot; } XN_i \to XS, \text{ sot, } i \to \infty,$$

$$f(N_i)X \to f(S)X, \text{ sot; } Xf(N_i) \to Xf(S), \text{ sot, } i \to \infty \quad ,$$

for an arbitrary X in B(H). This implies that:

$$N_iX - XN_i \to SX - XS, \text{ sot, } i \to \infty, \qquad (7)$$

$$f(N_i)X - Xf(N_i) \to f(S)X - Xf(S), \text{ sot, } i \to \infty \quad , \qquad (8)$$

In [6], we have, for every $\epsilon>0$, $N=D_\epsilon+K_\epsilon$ and $f(N)=f(D_\epsilon)+C_\epsilon$, where D_ϵ is a diagonal operator, $\|K_\epsilon\|_2 < \epsilon, \|C_\epsilon\|_2 \to 0$, as $\epsilon \to 0$.

If $D_\epsilon e_n=\lambda_n e_n$, and $X=(x_{ij})$, is the corresponding matrix of X, relative to the orthonormal basis (e_n) of H, then the (ij) entry for $D_\epsilon X-XD_\epsilon$ is $(\lambda_i-\lambda_j)x_{ij}$. Also the (ij) entry for $f(D_\epsilon)X-Xf(D_\epsilon)$ is $\left(f(\lambda_i)-f(\lambda_j)\right)x_{ij}$.

Hence,

$$\|(NX-XN)e_n\|-\|(D_\epsilon X-XD_\epsilon)e_n\| \le \|(K_\epsilon X-XK_\epsilon)e_n\| \le 2\|K_\epsilon\|_2 \|X\|,$$

and

$$\|(f(N)X-Xf(N))e_n\|-\|(f(D_\epsilon)X-Xf(D_\epsilon))e_n\| \le \|(C_\epsilon X-XC_\epsilon)e_n\| \le 2\|C_\epsilon\|_2 \|X\|.$$

Moreover,

$$\|(f(D_\epsilon)X-Xf(D_\epsilon))e_n\|^2 = \sum_{i=1}^{\infty}|f(\lambda_i)-f(\lambda_n)|^2|x_{in}|^2 \le k^2\sum_{i=1}^{\infty}|\lambda_i-\lambda_n|^2|x_{in}|^2 \le k^2\|(D_\epsilon X-XD_\epsilon)e_n\|^2.$$

Letting $\epsilon \to 0$, we obtain:

$$\|(f(N)X - Xf(N))e_n\| \le k\|(NX - XN)e_n\|, \qquad (9)$$

for every n.

Having obtained these results we have,

$$\|(SX - XS)e_n\| - \|(N_iX - XN_i)e_n\| \le \|[(SX - XS) - (N_iX - XN_i)]e_n\| \to 0$$

as $i \to \infty$.

Since $f(N_i)$ converges to $f(S)$ in (sot) one has:

$$\|(f(S)X - Xf(S))e_n\| - \|(f(N_i)X - Xf(N_i))e_n\| \le \|[(f(S)X - Xf(S)) - (f(N_i)X - Xf(N_i))]e_n\| \to 0$$

as $i \to \infty$.

Combining these inequalities, for arbitrary $\epsilon > 0$, e in H and sufficiently large i and using (9), one obtains:

$$\begin{aligned}\|(f(S)X - Xf(S))e_n\| &\le \|(f(N_i)X - Xf(N_i))e_n\| + \epsilon/2 \\ &\le k\|(N_iX - XN_i)e_n\| + \epsilon/2 \\ &< k\left\{\|(SX - XS)e_n\| + \frac{\epsilon}{2k}\right\} + \epsilon/2 \\ &< k\|(SX - XS)e_n\| + \epsilon .\end{aligned}$$

Thus (since ϵ is arbitrary),

$$\|(f(S)X - Xf(S))e_n\| \le \|(SX - XS)e_n\| , \qquad (10)$$

Therefore,

$$\begin{aligned}\|f(S)X - Xf(S)\|_2^2 &= \sum_{n=1}^{\infty}\|(f(S)X - Xf(S))e_n\|^2 \\ &\le \sum_{n=1}^{\infty} k^2\|(SX - XS)e_n\|^2 \\ &\le k^2\|SX - XS\|_2^2\end{aligned}$$

which ends the proof.

Remark: The analyticity of the Lipschitz function in Theorem 1 is essential, this is emphasized by the following.

Example: Let S be the unilateral shift on H, defined by $Se_n=e_{n+1}$, n=1,2,... and U be the bilateral shift on H, defined by $Ue_n=e_{n+1}$, n=..., -1, 0, 1, 2,... and let $f(z) = \bar{z}$. This function is Lipschitz with k=1, but not analytic on a neighborhood of $\sigma(S)$, which is the closed unit disc $\bar{D}$. It is known that S is subnormal (but not normal), with adjoint $S^* = f(S)=f(U)|_H=U^*|_H$, where U is the bilateral shift; a normal extension of S to $H\oplus H^{\perp}$, see [3]. If we assume that Theorem 1 is true

for non analytic Lipschitz function, and we let X=S, we would obtain that:

$$\|S^*S - SS^*\| \le \|SS - SS\| = 0,$$

which means that S is normal, which is a contradiction.

Corollary: Let S_1, S_2 be subnormal operators in B(H) and let $f \in R(E) \cap L(E)$, $E \supset \sigma(S_1) \cup \sigma(S_2)$ then:

$$\|f(S_1)X - Xf(S_2)\|_2 \le k\|S_1X - XS_2\|_2,$$

for some k>0, and every X in B(H). If X=I, then

$$\|f(S_1) - f(S_2)\|_2 \le k\|S_1 - S_2\|_2$$

Proof: Define S and Y on H⊕H by:

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}.$$

The operator S is subnormal and for any $f \in R(E)$,

$$f(s) = \begin{bmatrix} f(S_1) & 0 \\ 0 & f(S_2) \end{bmatrix},$$

Therefore, we have:

$$\|SY - YS\|_2 = \|S_1X - XS_2\|_2,$$

and

$$\|f(S)Y - Yf(S)\|_2 = \|f(S_1)X - Xf(S_2)\|_2.$$

Applying Theorem 1 to S, for $f \in R(E) \cap L(E)$, we obtain the conclusion, and the special case follows by taking X=I.

Theorem2: If S is a subnormal operator on H, whose dual is T on $H^{\perp}$ and *if* f is in $R(E) \cap L(E)$, where $\sigma(S) \subset E$, then,

$$\|f(S)X - Xf(T^*)\|_2 \leq k\|SX - XT^*\|_2,$$

for some $k>0$, and every $X \in B(H^{\perp}, H)$; the algebra of bounded linear operators from $H^{\perp}$ to H. In particular, if $H^{\perp}$ is a copy of H, (i.e., $H^{\perp} \cong H$), then:

$$\|f(S) - f(T^*)\|_2 \leq k\|S - T^*\|_2,$$

Proof: for $X \in B(H^{\perp},H)$, define $Y \in B(K)$, $K=H \oplus H^{\perp}$ by,

$$Y = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}.$$

Using the representations (1) one can show that:

$$NY - YN = \begin{bmatrix} S & A \\ 0 & T^* \end{bmatrix}\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S & A \\ 0 & T^* \end{bmatrix} = \begin{bmatrix} 0 & SX - XT^* \\ 0 & 0 \end{bmatrix}.$$

Similarly for $f \in R(E) \cap L(E)$, and using (2), one obtains:

$$f(N)Y - Yf(N) = \begin{bmatrix} 0 & f(S)X - Xf(T^*) \\ 0 & 0 \end{bmatrix}.$$

For an orthonormal basis (e_n) for $H^{\perp}$, we obtain:

$$(NY - YN)\begin{bmatrix} 0 \\ e_n \end{bmatrix} = \begin{bmatrix} (SX - XT^*)e_n \\ 0 \end{bmatrix}, \qquad (11)$$

and

$$(f(N)Y - Yf(N))\begin{bmatrix} 0 \\ e_n \end{bmatrix} = \begin{bmatrix} (f(S)X - Xf(T^*))e_n \\ 0 \end{bmatrix}. \qquad (12)$$

Combining (11), (12) and applying (9), one obtains:

$$\begin{aligned}\left\|\left(f(S)X - Xf(T^*)\right)e_n\right\| &= \left\|(f(N)Y - Yf(N))e_n\right\| \\ &\le k\left\|(NY - YN)e_n\right\| \\ &\le k\left\|(SX - XT^*)e_n\right\|,\end{aligned}$$

for every n, and thus:

$$\begin{aligned}\left\|f(S)X - Xf(T^*)\right\|_2^2 &= \sum_{n=1}^{\infty}\left\|(f(S)X - Xf(T^*))e_n\right\|^2 \\ &\le k^2\sum_{n=1}^{\infty}\left\|(SX - XT^*)e_n\right\|^2 \\ &\le k^2\left\|(SX - XT^*)e_n\right\|_2^2 .\end{aligned}$$

This implies that:

$$\left\|f(S)X - Xf(T^*)\right\|_2 \le k\left\|SX - XT^*\right\|_2,$$

for the other conclusion let X=I on $H^{\perp}=H$, and this proves the theorem.

Two operators T, S on H are said to be unitarily equivalent if there is a unitary operator u on H such that:

$$T=uSu^*.$$

For unitarily equivalent operators we prove the following:

Theorem3: Let $S\in B(H)$, $f\in R(E)\cap L(E)$, where $\sigma(S)\subset E$, such that

$$\|f(S)X - Xf(S)\|_2 \le k\|SX - XS\|_2 \qquad (13)$$

for some $k>0$, and any $X\in B(H)$. If $T\in B(H)$ is unitarily equivalent to S, then

$$\|f(T)Y - Yf(T)\|_2 \le k\|TY - YT\|_2,$$

for some $k>0$, and any $Y\in B(H)$.

Proof: Let u be a unitary operator on H, such that $T=uSu^*$. Since $f\in R(E)$, it is not difficult to show that $f(T)=uf(S)u^*$. If (e_n) is an orthonormal basis for H, it is known that (u^*e_n) is an orthonormal basis for H too, for any unitary operator u. Thus one has

$$\begin{aligned}\|(f(T)Y - Yf(T))e_n\| &= \|(uf(S)u^*Y - Yuf(S)u^*)e_n\| \\ &= \|(uf(S)u^*Yuu^* - uu^*Yuf(S)u^*)e_n\| \\ &= \|(uf(S)X - Xf(S))u^*e_n\| \\ &= \|(f(S)X - Xf(S))g_n\|, \qquad (14)\end{aligned}$$

where

$X=u^*Yu$, $Y\in B(H)$, $(g_n)=(u^*e_n)$.

Using (14), we have:

$$\begin{aligned}\|f(T)Y - Yf(T)\|_2^2 &= \sum_{n=1}^{\infty} \|(f(T)Y - Yf(T))e_n\|^2 \\ &= \sum_{n=1}^{\infty} \|(f(S)X - Xf(S))g_n\|^2 \\ &= \|f(S)X - Xf(S)\|_2^2 \ .\end{aligned}$$

By the assumption of the theorem one concludes that

$$\begin{aligned}\|f(T)Y - Yf(T)\|_2^2 &\le k^2 \|SX - XS\|_2^2 \\ &\le k^2 \sum_{n=1}^{\infty} \|(SX - XS)g_n\|^2 \\ &\le k^2 \sum_{n=1}^{\infty} \|u(SX - XS)u^* e_n\|^2 \\ &\le k^2 \sum_{n=1}^{\infty} \|(uSu^* uXu^* - uXu^* uSu^*)e_n\|^2 \\ &\le k^2 \sum_{n=1}^{\infty} \|(TY - YT)e_n\|^2 \\ &= k^2 \|(TY - YT)\|_2^2 \ ,\end{aligned}$$

which shows that

$$\|f(T)Y - Yf(T)\|_2 \le k \|TY - YT\|_2 \ .$$

To end up the proof we should remark that since T is unitarily equivalent to S, they have the same spectrum, so that $\sigma(T)\subset E$ as well.

Finally we end this article by the following.

Theorem:4: If S satisfies (13), then every operator in the closure of the unitary orbit u(S) of S in the uniform operator topology (uot) satisfies (13), where

$$u(S)=\{uSu^{*}: u \text{ is unitary on } H\}.$$

Proof: Since $S\in B(H)$ such that for $f\in R(E)\cap L(E)$, we have

$$\|f(S)X - Xf(S)\|_2 \leq k\,\|SX - XS\|_2, \qquad (15)$$

for some $k>0$, and every X in B(H). Theorem 3 shows that every operator in u(S) satisfies (13). So that, we assume that (u_n) is a sequence of unitary operators on H, such that $(u_nSu^{*}_n)$ converges to A in (uot), and we want to show that A satisfies (13). Indeed, from uniform convergence: $\|u_nSu_n^{*} - A\| \to 0$, we have $(u_nf(S)u^{*}_n)$ converges to f(A) in (uot), for every $f\in R(E)$. By the upper semi continuity of the spectrum, one has

$$\sigma(f(A)) \subset \sigma(u_n f(S) u^*_n) \subset E.$$

Since uniform operator topology convergence of (T_i) to A implies uniform operator topology convergence of $f(T_i)$ to $f(A)$, for $f \in R(E)$, one concludes for unitary (u_n) that

$$(u_n f(S) u^*_n) = f(u_n S u^*_n),$$

converges to f(A) in (uot). Now we have

$$\begin{aligned}
\|(f(A)X - Xf(A))e\| &\le \|(f(A)X - Xf(A)) - ((u_n f(S)u_n^*)X - X(u_n f(S)u_n^*))e\| \\
&\quad + \|((u_n f(S)u_n^*)X - X(u_n f(S)u_n^*))e\| \\
&\le \epsilon/2 + \|u_n(f(S)Y_n - Y_n f(S)u_n^*)e\| \\
&\le \epsilon/2 + \|(f(S)Y_n - Y_n f(S))g\| \\
&\le \epsilon/2 + k\|(SY_n - Y_n S)g\| \\
&\le \epsilon/2 + k\|(u_n S u_n^* X - X u_n S u_n^*)e\| \\
&\le \epsilon/2 + k\|(u_n S u_n^* X - X u_n S u_n^*)e - (AX - XA)e\| \\
&\quad + k\|(AX - XA)e\| \\
&\le \epsilon/2 + .k\left(\frac{\epsilon}{2k}\right) + k\|(AX - XA)e\| \\
&\le \epsilon + k\|(AX - XA)e\| .
\end{aligned}$$

Since ϵ is arbitrary, we have

$$\|(f(A)X - Xf(A))e\| \le k\|(AX - XA)e\|.$$

Now, for an orthonormal basis (e_n) for H,

$$\begin{aligned}\|f(A)X - Xf(A)\|_2^2 &= \sum_{n=1}^{\infty}\|(f(A)X - Xf(A))e_n\|^2 \\ &\le k^2\sum_{n=1}^{\infty}\|(AX - XA)e_n\|^2 \\ &\le k^2\|AX - XA\|_2^2 ,\end{aligned}$$

and thus A satisfies (13) for every A in the uniform closure of u(S).

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