An-Najah Univ. J. Res. (N. Sc), Vol. 16(2), 2002

Composition Operators on Orlicz and Bochner Spaces

المؤثرات المركبة على فضاءات أورلكس وبوخنر

Mahmud Masri

Mathematics Department, Faculty of Science, An-Najah National University, Nablus, Palestine.

Received: (16/9/2000), Accepted: (7/5/2001)

Abstract

Let (T, M, μ) be a finite positive measure space, X a Banach space, ϕ a modulus function and $f: T \to X$ a strongly measurable function. The Orlicz space is $L^{\phi}(\mu, X) = \left\{ f : \int_T \phi(\|f(t)\|) d\mu(t) < \infty \right\}$. The space of Bochner p-integrable functions, $1 \le p < \infty$, is $L^p(\mu, X) = \left\{ f : \int_T \|f(t)\|^p d\mu(t) < \infty \right\}$. Also $L^{\infty}(\mu, X) = \left\{ f : \operatorname{ess\,sup}_{t \in T} \|f(t)\| < \infty \right\}$. When $\phi(x) = x^p$, $0 < p \le 1$, $L^{\phi}(\mu, X) = L^P(\mu, X)$. Let $\Psi : T \to T$ be a function with $\Psi^{-1}(A) \in M$ for all $A \in M$ and define $C_{\Psi}(f) = f \circ \Psi$. We prove that C_{Ψ} is a bounded linear operator on $L^{\phi}(\mu, X)$ and $L^p(\mu, X)$, $0 < p \le \infty$, when $\frac{d\mu_{\Psi}}{d\mu} \in L^{\infty}(\mu, \mathbf{C})$ where $\mu_{\Psi}(A) = \mu(\Psi^{-1}(A))$ for all $A \in M$ and **C** is the complex numbers.Also, we show that C_{Ψ} is an isometry of $L^{\phi}(\mu, X)$ and $L^p(\mu, X)$, $0 < p < \infty$ iff $\frac{d\mu_{\Psi}}{d\mu} = 1$ a.e. .Moreover, C_{Ψ} is an isometry of $L^{\infty}(\mu, X)$ iff $\mu << \mu_{\Psi}$. This generalizes some previous results of the special case $L^p(\mu, C)$ and proves similar results for $L^{\phi}(\mu, X)$.

ملخص

ليكن (T, M, μ) فضاءا قياسيا موجبا ومنتهيا و X هو فضاء بناخ و ϕ اقتران مطلق القيمة و $f: T \to X$ هو اقتران قياسي بقوة. فان فضاء أورلكس هو $L^{\phi}(\mu, X) = \left\{ f : \int_T \phi(\|f(t)\|) d\mu(t) < \infty \right\}$، وفضاء بوخنر

عندما $1 \le p < \infty$ هو $L^p(\mu, X) = \left\{ f : \int_T \|f(t)\|^p \, d\mu(t) < \infty \right\}$. أيضا $L^\infty(\mu, X) =$

$\left\{ f : \operatorname{ess\,sup}_{t \in T} \|f(t)\| < \infty \right\}$. وعندما $\phi(x) = x^p$ ، فانّ $0 < p \le 1$

$L^\phi(\mu, X) = L^P(\mu, X)$.

ليكن $\Psi : T \to T$ هو اقتران بحيث ان $\Psi^{-1}(A) \in M$ لكل $A \in M$. نعرّف $C_\Psi(f) = f \circ \Psi$.

في هذا البحث سنثبت ،ان شاء الله، أنّ C_Ψ هو مؤثرا خطيا ومحدودا على كل من $L^\phi(\mu, X)$ و $L^p(\mu, X)$ ،

عندما $0 < p \le \infty$ حيث أنّ $\frac{d\mu_\Psi}{d\mu} \in L^\infty(\mu, \boldsymbol{C})$ لكل $A \in M$ $\mu_\Psi(A) = \mu(\Psi^{-1}(A))$

و $\boldsymbol{C}$ هي الاعداد العقدية. *ايضا* سنثبت ،ان شاء الله، أنّ C_Ψ مؤثرا تقايسيا على $L^\phi(\mu, X)$ و $L^p(\mu, X)$ ،

اذا وفقط اذا $0 < p < \infty$ a.e. $\frac{d\mu_\Psi}{d\mu} = 1$. اما بالنسبة ل $L^\infty(\mu, X)$ فان C_Ψ يكون تقايسيا اذا وفقط

اذا $\mu << \mu_\Psi$. وهذا يعتبر تعميما لنتائج الحالة الخاصة $L^p(\mu, \boldsymbol{C})$ ويثبت نتائج مماثلة في $L^\phi(\mu, X)$.

1. Introduction:

If ϕ is a strictly increasing continuous subadditive function on $[0, \infty)$ and satisfies $\phi(x) = 0$ iff x = 0, then we call ϕ a modulus function. Let (T, M, μ) be a finite positive measure space, i.e., T is a set, M is a $\sigma -$ algebra and μ is a positive measure with $\mu(T) < \infty$. If X is a Banach space, then a function $s : T \to X$ is called a simple function if its range contains finitely many points $x_1, x_2, ..., x_n$ and $E_i = s^{-1}(\{x_i\})$, $i = 1,2,...n$ are measurable sets. Such a function s can be written as $s = \sum_{i=1}^{n} x_i \chi_{E_i}$, where χ_{E_i} is the characteristic function of the set E_i and $E_i \cap E_j = \Phi, \text{for} i \ne j, i, j = 1, 2, ..., n.$ A function $f : T \to X$ is said to be strongly measurable if there exists a sequence $\{s_n\}$ of simple functions such that

$$\lim_{n \to \infty} \|s_n(t) - f(t)\| = 0 \text{ a.e.}$$

The Orlicz space $L^{\phi}(\mu, X)$ is the set of all (equivalence classes) of strongly measurable functions f with

$$\|f\|_{\phi} = \int_T \phi(\|f(t)\|) d\mu(t) < \infty .$$

If for all $f, g \in L^{\phi}(\mu, X)$ we define $d(f, g) = \|f - g\|_{\phi}$, then d is a metric on $L^{\phi}(\mu, X)$ under which it becomes a complete topological vector space [1,p.70]. For $1 \le p < \infty$, $L^{p}(\mu, X)$ denotes the Banach space of (equivalence classes of) strongly measurable functions f such that $\int_T \| f(t) \|^{\mathrm{P}} \, d\mu(t) < \infty$. The norm in $L^{p}(\mu, X)$ is given by

$$\|f\|_{p} = \left(\int_T \|f(t)\|^{p} d\mu(t) \right)^{\frac{1}{p}}$$

The essentially bounded strongly measurable functions f form Banach space $L^{\infty}(\mu, X)$ with norm given by $\|f\|_{\infty} = ess \sup_{t \in T} \|f(t)\|$.

If ϕ is the modulus function $\phi(x) = x^{p}$, $0 < p \le 1$, then $L^{\phi}(\mu, X)$ is the space $L^{p}(\mu, X)$. Since [2, p. 159], for any modulus function ϕ, $\limsup_{x \to \infty} \frac{\phi(\mathrm{x})}{\mathrm{x}} \le \phi(1)$, it follows that $L^{1}(\mu, X) \subseteq L^{\phi}(\mu, X)$.

For simplicity of notation we write $L^{p}(\mu, C) = L^{p}, 0 < p \le \infty$, $L^{\phi}(\mu, C) = L^{\phi}$.

Also, $\| \ \|_{p} = | \ |_{p}$, $\| \ \|_{\infty} = | \ |_{\infty}$, $\| \ \|_{\phi} = | \ |_{\phi}$ when X is the complex numbers $\boldsymbol{C}$.

We mean by a measurable transformation on T a function $\Psi : T \to T$ such that $\Psi^{-1}(A) \in M$ for all $A \in M$. It is easy to see that Ψ induces a positive measure μ_Ψ on M where $\mu_\Psi(A) = \mu(\Psi^{-1}(A))$ for all $A \in M$. Also, Ψ induces a composition operator C_Ψ on strongly measurable functions given by $C_\Psi(f) = f \circ \Psi$ when μ is complete or $\mu_\Psi << \mu$, i.e. , μ_Ψ is absolutely continuous with respect to μ. It is known that [3,p.122] $C_\Psi(L^\infty) \subseteq L^\infty$, $\|C_\Psi\| \le 1$, and C_Ψ is an $L^\infty -$isometry iff $\mu << \mu_\Psi$ also . Moreover, $C_\Psi(L^p) \subseteq L^p, 1 \le p < \infty$ iff $\left|\frac{d\mu_\Psi}{d\mu}\right|_\infty < \infty$. In this paper we prove similar results for C_Ψ on $L^p(\mu, X)$ and $L^\phi(\mu, X)$ for any Banach space X and give necessary and sufficient conditions for C_Ψ to be an isometry.

2. Composition Operators:

Let (T, M, μ) be a finite positive measure space, X a Banach space and $\Psi : T \to T$ a measurable transformation. The induced composition operator C_Ψ satisfies the following result**.**

Proposition 2.1

If $\mu_\Psi << \mu$, then $C_\Psi(f) = f \circ \Psi$ is strongly measurable for every strongly measurable function $f : T \to X$.

Proof

Obviously for $x \in X$ and $A \in M$ we have $C_\Psi(x\chi_A) = x\chi_{\Psi^{-1}(A)}$. Thus if $\{s_n\}$ is a sequence of simple functions such that $\lim_{n\to\infty} \|s_n(t) - f(t)\| = 0$ a.e.,

then $\{C_\Psi(s_n)\}$ is a sequence of simple functions such that $\lim_{n\to\infty}\|(C_\Psi(s_n))(t)-(C_\Psi(f))(t)\|=0$ a.e. since $\mu_\Psi << \mu$.

We note that Proposition 2.1 is true if the condition ($\mu_\Psi << \mu$) is replaced by "μ is complete" [4, p.114]. In this case any strongly measurable function is measurable in the classical sense, i.e., the inverse image of every open set is measurable.

We call C_Ψ an isometry of $L^p(\mu,X), 0<p\le\infty$ if $\|C_\Psi f\|_p = \|f\|_p$ for all $f\in L^p(\mu,X)$ and C_Ψ is an isometry of $L^\phi(\mu,X)$ if $\|C_\Psi f\|_\phi = \|f\|_\phi$ for all $f\in L^\phi(\mu,X)$. The following results are extensions of those in [3, p. 122] from $\boldsymbol{C}$ to any Banach space *X*.

Theorem 2.2

$C_\Psi(L^\infty(\mu,X))\subseteq L^\infty(\mu,X)$, $\|C_\Psi\|\le 1$ and C_Ψ is an isometry of $L^\infty(\mu,X)$ iff $\mu << \mu_\Psi$.

Proof

Let $f\in L^\infty(\mu,X)$. Since $\mu_\Psi << \mu$ it is easily seen that $\|f(t)\|\le\|f\|_\infty<\infty$ a.e. implies that $\|C_\Psi f\|_\infty \le \|f\|_\infty$. Thus $C_\Psi(L^\infty(\mu,X))\subseteq L^\infty(\mu,X)$, and $\|C_\Psi\|\le 1$. Suppose C_Ψ is an isometry of $L^\infty(\mu,X)$. For $x\in X$, $x\neq 0$ and $A\in M$, we certainly have $\mu(A)=0$ iff $\|x\chi_A\|_\infty = 0$. Since

$$\|x.\chi_{\Psi^{-1}(A)}\|_\infty = \|C_\Psi(x.\chi_A)\|_\infty = \|x\chi_A\|_\infty$$

it follows that $\mu(A)=0$ whenever $\mu_\Psi(A)=\mu((\Psi^{-1}(A))=0$. Thus $\mu << \mu_\Psi$.

Conversely, suppose $\mu << \mu_\Psi$. From above, it suffices to show that $\|f\|_\infty \le \|C_\Psi(f)\|_\infty$ for all $f \in L^\infty(\mu, X)$. Let $f \in L^\infty(\mu, X)$. Then

$$\|(C_\Psi(f))(t)\| \le \|C_\Psi(f)\|_\infty \le \|f\|_\infty < \infty$$

for all $t \notin A$ for some $A \in M$ with $\mu(A) = 0$. Hence, $\|f(s)\| \le \|C_\Psi(f)\|_\infty$ for all $s \in \Psi(A^c)$ where $A^c = T - A$. Since $\mu_\Psi(E) = 0$ iff $\mu(E) = 0$ for all $E \in M$ and $\Psi^{-1}((\Psi(A^c))^c) \subseteq A$ it follows that $0 = \mu(A) = \mu_\Psi((\Psi(A^c))^c)$ when μ is complete, i.e., any subset of a set of measure zero is measurable. If μ is not complete it can be replaced by its completion (see[5,p.29]). Therefore, $\|f(s)\| \le \|C_\Psi(f)\|_\infty < \infty$ a.e.. Thus $\|f\|_\infty \le \|C_\Psi(f)\|_\infty$ for all $f \in L^\infty(\mu, X)$ and C_Ψ is an isometry of $L^\infty(\mu, X)$.

Theorem 2.3

Let (T, M, μ) be a finite positive measure space, $\Psi : T \to T$ a measurable transformation, $1 \le p < \infty$, and $\mu_\Psi << \mu$. Then

a. $C_\Psi(L^p(\mu, X)) \subseteq L^p(\mu, X)$ if $\dfrac{d\mu_\Psi}{d\mu} \in L^\infty$

b. C_Ψ is an isometry of $L^p(\mu, X)$ iff $\dfrac{d\mu_\Psi}{d\mu} = 1$ a.e.

Proof

a. Let $\dfrac{d\mu_\Psi}{d\mu} \in L^\infty$. Then [6,p.164] for all $f \in L^p(\mu, X)$ we have

$$\|C_\Psi(f)\|_p^p = \int_T \|f(\Psi(t))\|^p \, d\mu(t) = \int_T \|f(t)\|^p \left(\frac{d\mu_\Psi}{d\mu}\right)(t) d\mu(t) \quad \text{.....(1)}$$

Thus (1) gives

$$\left\|C_\Psi(f)\right\|_p \le \left|\frac{d\mu_\Psi}{d\mu}\right|_\infty^{\frac{1}{p}} \left\|f\right\|_p \quad \text{..... ....(2)}$$

for all $f \in L^p(\mu, X)$. Therefore, C_Ψ is a bounded linear operator on $L^p(\mu, X)$ and $\left\|C_\Psi\right\| \le \left|\frac{d\mu_\Psi}{d\mu}\right|_\infty^{\frac{1}{p}}$.

b. Suppose C_Ψ is an isometry of $L^p(\mu, X)$, $1 \le p < \infty$. For each $f \in L^p$ define $\tilde{f}(t) = f(t)\frac{x}{\|x\|}$ for all $t \in T$, where $x \in X$ and $x \neq 0$. Then $\tilde{f} \in L^p(\mu, X)$ and $\left|C_\Psi(f)\right|_p = \left\|C_\Psi(\tilde{f})\right\|_p = \left\|\tilde{f}\right\|_p = \left|f\right|_p$. Thus C_Ψ is an isometry of L^p and hence [3] implies that $\frac{d\mu_\Psi}{d\mu} \in L^\infty$. Next, let $f(t) = \frac{x}{\|x\|}$ for all $t \in T$, where $x \in X$ and $x \neq 0$. Then $f \in L^p(\mu, X)$ and (2) implies that $\left|\frac{d\mu_\Psi}{d\mu}\right|_\infty \ge 1$. Also, for this f by [6,p.164] we get

$$\int_T d\mu(t) = \left\|f\right\|_p^p = \left\|C_\Psi(f)\right\|_p^p = \int_T \left\|f(\Psi(t))\right\|^p d\mu(t) = \int_T \left(\frac{d\mu_\Psi}{d\mu}\right)(t)\, d\mu(t)$$

Therefore, $\int_T \left(\left(\frac{d\mu_\Psi}{d\mu}\right)(t) - 1\right) d\mu(t) = 0$ implies that $\frac{d\mu_\Psi}{d\mu} = 1$ a.e

The converse is clear from (1).

The next results deal with C_Ψ on $L^\phi(\mu, X)$.

Theorem 2.4

Let ϕ be a modulus function. Then

a. C_Ψ is a bounded linear operator on $L^\phi(\mu, X)$ if $\frac{d\mu_\Psi}{d\mu} \in L^\infty$.

b. C_Ψ is an isometry of $L^\phi(\mu, X)$ iff $\frac{d\mu_\Psi}{d\mu} = 1$ a.e.

Proof

a. For $f \in L^\phi(\mu, X)$ by [6,p.164] we have

$$\|C_\Psi(f)\|_\phi = \int_T \phi\left(\|f(\Psi(t))\|\right) d\mu(t) = \int_T \phi\left(\|f(t)\|\right)\left(\frac{d\mu_\Psi}{d\mu}\right)(t)\, d\mu(t) \quad \text{.....(3)}$$

Thus (3) gives

$$\|C_\Psi(f)\|_\phi \le \left|\frac{d\mu_\Psi}{d\mu}\right|_\infty \|f\|_\phi \quad \text{..... ....(4)}$$

for all $f \in L^\phi(\mu, X)$. Therefore, C_Ψ is a bounded linear operator on $L^\phi(\mu, X)$ and $\|C_\Psi\| \le \left|\frac{d\mu_\Psi}{d\mu}\right|_\infty$.

b. Suppose C_Ψ is an isometry of $L^\phi(\mu X)$. For $f \in L^1$ let $\tilde{f}(t) = \phi^{-1}\left(|f(t)|\right)\frac{x}{\|x\|}$ for all $t \in T$, where $x \in X$ and $x \neq 0$. Then $|C_\Psi(f)|_1 = \left\|C_\Psi(\tilde{f})\right\|_\phi = \left\|\tilde{f}\right\|_\phi = |f|_1$. Therefore, C_Ψ is an isometry of L^1 and hence by [3] $\frac{d\mu_\Psi}{d\mu} \in L^\infty$.

Moreover, for a nonzero $x \in X$ and $f(t) = \frac{x}{\|x\|}$ for all $t \in T$, by (4) one has $\left|\frac{d\mu_\Psi}{d\mu}\right|_\infty \geq 1$. Also, for such f by [6] we have

$$\int_T \phi(1) d\mu(t) = \|f\|_\phi = \|C_\Psi f\|_\phi = \int_T \phi(1)\left(\frac{d\mu_\Psi}{d\mu}\right)(t) d\mu(t)$$

This implies that $\frac{d\mu_\Psi}{d\mu} = 1$ a.e. Finally, the converse follows from (3).

Corollary 2.5

C_Ψ is an isometry of $L^p(\mu, X)$, $1 \leq p < \infty$, iff C_Ψ is an isometry of $L^\phi(\mu, X)$.

Proof

Clear from theorem 2.4 and theorem 2.5.

Finally, we note that if $\mu_\Psi << \mu$ then C_Ψ is $1-1$ on $L^p(\mu, X)$, $0 < p \leq \infty$, and $L^\phi(\mu, X)$ for any modulus function ϕ. Moreover, if Ψ is invertible with inverse Ψ^{-1}, then so is C_Ψ with inverse $C_{\Psi^{-1}}$.

References

1] Deeb, W. and Khalil, R., "Best approximation in $L^p(\mu, X)$, $0 < p < 1$", *J. Approx. Theory*, **58,** (1989), 68-77.

2] Deeb, W., "Multipliers and Isometries of Orlicz Spaces", *Proceedings of the conference on Mathematical Analysis and its Applications, Kuwait Univ.* Kuwait, Feb., (1985), 18-21

3] William C. Ridge, "Spectrum of a Composition Operator", *Proceedings of the A.M.S.,* **37(1)**, (1973), 121-127.

4] Light, W. and Cheney, W., "Approximation Theory in Tensor Product Spaces", Lecture Notes in Math. 1169, springler-verlag, Berlin, (1985).

5] Rudin, W., "Real and Complex Analysis", 2nd ed., McGraw-Hill, New York, (1974).

6] Halmos, P. R., "Measure Theory", Van Nostrand, Princeton, N.J., (1950).