MATHEMATICAL ANALYSIS OF A VIBRATING RIGID WATER TANK

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ملخص

البحث التالي يتناول حساب الضغط الهيدروديناميكي المؤثر على جدار خزان اسطواني الشكل بطريقة رياضية قصيرة ومبسطة نسبياً . ويتضح في هذا السياق بأن الحل رغم قصره يؤدي الى مركبة ضغط تعتمد على ذبذبة الحركة وأخرى مستقلة عنها .

Abstract

The hydrodynamic pressure distribution on the wall of a vibrating water tank is traditionally expressed as a summation of two components ; an impulsive component and a convective one obtained by separating the potential function into two parts . This requires solving Laplace's equation in two stages each with a separate set of boundry conditions .

The following is one step systematic solution to the problem in a frame moving with the tank . It proves to be simple , compact and could lead to the impulsive , frequency independent and the convective , frequency dependent components of pressure at the water tank wall .

Introduction

Water tanks are generally constructed of either reinforced concrete or of steel . As such , they are treated as rigid or flexible depending on the construction material and on the tank's relative dimensions . Dynamic analysis of such water – tank systems has traditionally been approached by separating the potential function into two parts : impulsive which is frequency independent , and convective , which is proportional to the sloshing frequency of the liquid . Although the intention is understood to be the separation of a frequency dependent component

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of the pressure from the frequency independent component , this requires the tedious task of having to solve the Laplace's equation twice each time with a different set of boundry conditions [7] .

This presentation illustrates a systematic solution to Laplace's equation to yield the total potential for the case of a rigid tank . It is noticed that by dropping the surface effects , the solution easily leads to the impulsive and the convective components , if that is desired . The present study is limited to the case of a rigid water tank subject to a harmonic ground excitation of unit amplitude and a steady state response .

Problem Statement and Its Solution

Consider the vertical cylindrical tank shown in Figure 1 . The tank is assumed rigid and subject to horizontal harmonic ground excitation of unit amplitude . For all practical purposes , the water in the tank may safely be assumed incompressible and its motion irrotational . Let the potential function representing the fluid motion be Φ . Then Laplace's equation of motion will be

$$\nabla^2 \Phi = 0 \tag{1}$$

The following are the boundary conditions written in a reference frame moving with the tank .

$$\frac{\partial \Phi}{\partial r} = 0 \quad \text{at} \quad r = r_0 . \tag{2}$$

and

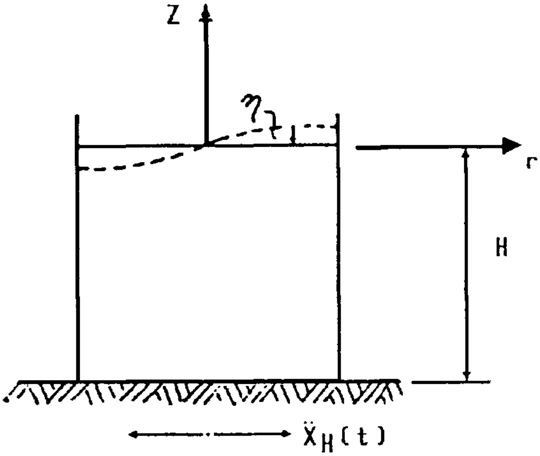
$$\frac{\partial \Phi}{\partial z} = 0 \quad \text{at} \quad z = -H \tag{3}$$

At the water surface , two conditions are available . One requires a particle on the surface to stay on the surface , and the other requires the pressure at the surface to have a prescribed value .

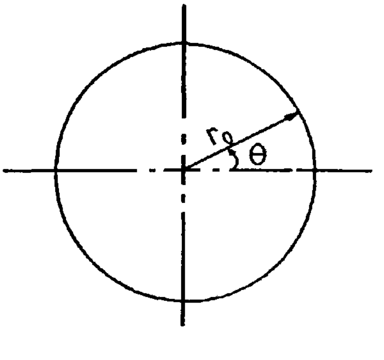
With reference to a moving frame , the Bernoulli's equation for the pressure distribution is

$$P = -\rho\left\{\frac{\partial \Phi}{\partial t} + \ddot{X}_H(t)\, r \cos\theta + gz \right\} \tag{5}$$

The first two terms in Equation 5 represent the dynamic pressure distribution while the third term is simply the hydrostatic pressure .



(a) Vertical section



(b) Horizontal section

Figure 1 . Rigid Water Tank Subjected to Seismic Excitation

Equation 1 is solved by the standard method of separation of variables. In cylindrical coordinates let

$$\Phi\ (z, r, \theta, t)\ =\ Z(z)\,.\,R(r)\,.\,\Theta(\theta)\,.\,\dot{T}(t) \qquad (6)$$

By inspection ,the eigenfunction in θ is found to be

$$\Theta(\theta)\ =\ \cos\theta \qquad (7)$$

This implies that the eigenfunctions in z and r are obtained from the solutions to the following two equations .

$$\frac{Z''(z)}{Z(z)}\ =\ k^2. \qquad (8)$$

and

$$r^2R'' + rR' + (k^2r^2 - 1)R = 0 \qquad (9)$$

Equation 8 is a well posed Sturm-Liouville problem with the following homogeneous Neumann boundary condition[4] .

$$\frac{dZ}{dz} = 0 \qquad \text{at } z = -H \qquad (10)$$

Equation 9 is a Bessel's equation governed by the boundary condition

$$= 0 \qquad \text{at } r = r_0 \qquad (11)$$

k^2 is to take all possible values (i . e . positive , zero and negative) for $k^2 \leq 0$ trivial solutions are obtained , however , for $k^2 > 0$ the solution is

$$Z(z)\ =\ C\cosh k\,(z + H) \qquad (12)$$

The standard solution to equation 9 is

$$R(r)\ =\ D\,J_1(kr)\ +\ E\,Y_1(kr) \qquad (13)$$

where J_1 is the Bessel function of the first kind and first order
Y_1 is the Bessel function of the second kind and first order

C, D, and E are constants of integration .

Since Y_1 leads to a singularity at the origin , it is dropped and the solution reduces to

$$R(r) = D J_1(kr) \tag{14}$$

Boundary condition 11 implies that either D = 0 which is a trivial solution or $J'_1(kr_0) = 0$. Hence , the eigenvalues are obtained as the roots to the equation .

$$J'_1(kr_0) = 0 \tag{15}$$

Because Equation 15 has an infinite number of roots , the solution is written as a summation of the following infinite series

$$\Phi = \cos\theta \sum_{n=1}^{\infty} \dot{T}_n(t) \cosh k_n(z + H) J_1(k_n r) \tag{16}$$

Where $\dot{T}_n(t)$ is a function of time yet to be determined . This is achieved by invoking the boundary conditions at the surface .

Prescribing the pressure at the surface $z = \eta$ to be atmospheric gives

$$\cos\theta \sum_{n=1}^{\infty} \ddot{T}_n(t) \cosh(k_n H) J_1(k_n r) + \ddot{X}_H(t) r \cos\theta + g\eta = 0 \tag{17}$$

in which η is unknown , but could be obtained from the following linearized kinematic boundary condition

$$\frac{\partial \eta}{\partial t} = \frac{\partial \Phi}{\partial z} \quad \text{at } z = 0$$

or

$$\eta = \int \frac{\partial \Phi}{\partial z} \, dt \qquad \text{at } z = 0 \tag{18}$$

From which it follows that

$$\eta = \cos\theta \sum_{n=1}^{\infty} T_n(t) k_n \sinh(k_n H) J_1(k_n r) \tag{19}$$

substituting for η into Equation 17 yields (20)

$$\sum_{n=1}^{\infty} \ddot{T}_n(t) \cosh(k_n H) J_1(k_n r) + g \sum_{n=1}^{\infty} T_n(t) k_n \sinh(k_n H) J_1(k_n r) + \ddot{X}_H(t) r = 0$$

Multiplying both sides of Equation 20 by $rJ_1(k_s r)$ and invoking the following orthogonality property of J_1.

$$\int_0^{r_o} r\, J_1(k_n r)\, J_1(k_s r) = 0\,, \text{ for } n \neq s$$

yields the following system of uncoupled linear differential equations for T_n.

$$\ddot{T}_n(t) + g\,k_n \tanh(k_n H)\, T_n(t) = \frac{-2r_o}{[(k_n r_o)^2 - 1]} \frac{\ddot{X}_H(t)}{\cosh(k_n H)\, J_1(k_n r_o)} \tag{21}$$

which is of the form

$$\ddot{T}_n(t) + B_n T_n(t) = C_n \ddot{X}_H(t) \tag{22}$$

where $B_n = g\,k_n \tanh(k_n H)$

$$C_n = \frac{-2r_o}{[(k_n r_o)^2 - 1]} \frac{1}{\cosh(k_n H)\, J_1(k_n r_0)}$$

For $\ddot{x}_H(t) = e^{-\omega t}$, the particular solution to equation 22 is

$$T_n(t) = \frac{C_n e^{-i\omega t}}{B_n - \omega^2} \tag{23}$$

with all of the eigenfunctions computed , the potential function takes the form

$$\Phi = -\cos\theta(i\omega) \sum_{n=1}^{\infty} \frac{C_n e^{-i\omega t}}{B_n - \omega^2} \cosh k_n(z + H)\, J_1(k_n r) \tag{24}$$

Now that the potential function is fully determined , the water surface displacement and the pressure distribution at the water tank wall may easily be computed for any value of excitation frequency . The dynamic pressure is obtained from Bernoulli's equation

$$P = \rho \cos\theta \; e^{-i\omega t} \sum_{n=1}^{\infty} \frac{C_n \, \omega^2}{B_n - \omega^2} \cosh k_n (z + H) J_1 (k_n r_o) - \rho r_o \cos\theta \; e^{-i\omega t} \qquad (25)$$

Figure 2 is a plot of the amplitude of the dynamic pressure distribution ($\bar{p}$) on the wall of a tank having a 10−meter radius and water 5 meters in depth . The computations are for $\theta = 0$ and $\omega = 4.42$ rad/second . This value of ω is randomly chosen to be different from any natural frequency of the sloshing fluid

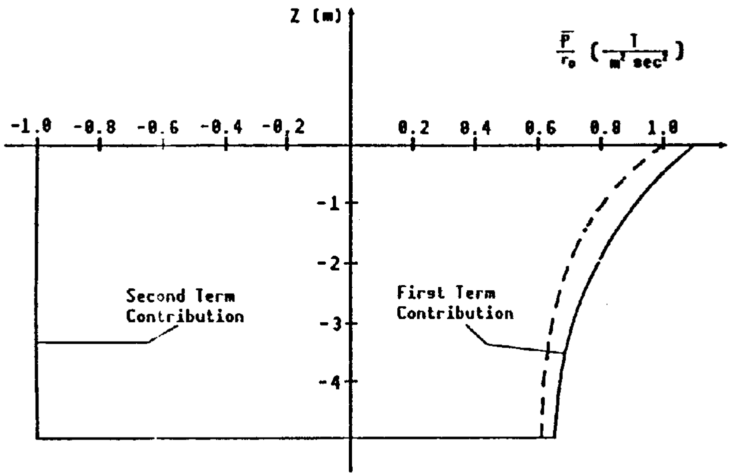


Figure 2 . Pressure distribution at the wall taken for $\theta = 0$ ($\omega = 4.42$, – – – – – – – – limiting value)

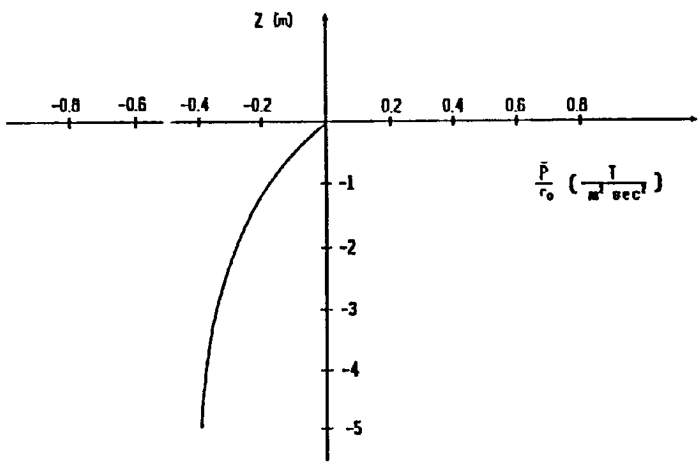


Figure 3 .The impulsive pressure distribution taken for $\theta = 0$

Discussion

Upon the examination of Equation 25 and of Figure 2 , it is noticed that while the first term is frequency and depth dependent , the second is proportional to the radius only . Although the contribution of the first term quickly reaches a limiting value as the value of ω is increased yet it does not look appropriate to neglect it under any circumstance . Moreover , since in the analysis of fluid – structure interaction systems , it is convenient to deal with a frequency independent pressure distribution ; all that is needed is to set $\frac{g}{\omega^2}=0$ in equation 25 . The result is shown in Figure 3. This is exactly what is referred to in the literature [5] as the impulsive pressure . This operation means that the surface effects are neglected which is universally accepted at rather high values of excitation frequencies .

List of Symbols

Φ = potential function of fluid motion

r, z, θ = cylindrical coodinate system.

ρ = fluid density

$\ddot{X}_H$ = horizontal ground acceleration

t = time

ω = angular frequency of oscillation

r_o = radius of tank

H = depth of fluid

k = wave number

η = wave height

$\bar{p}$ = pressure amplitude

J_1 = the Bessel function of the first kind and first order

Y_1 = the Bessel function of the second kind and first order

Other symbols are defined as they appear in text .

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