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Strongly Singular Potentials in One Dimension

جهود قوية القطبية في بعد واحد

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Abstract:

Strongly singular potentials in one dimension are analyzed, namely the n[th] derivative interaction $\delta^{(n)}(x)$. For the repulsive case, the reflection and transmission coefficients are derived. It is shown that for n = even, these coefficients satisfy the unitarity of the scattering matrix for all values of the incident energy and the barrier becomes perfectly reflective for a specific value of the energy. For n=odd $\neq 1$, however, the coefficients do not satisfy the unitarity of the scattering matrix except for one specific value of the incident energy at which the barrier becomes perfectly reflective. For the n=1 case, the results showed that the coefficients satisfy the unitarity of the scattering matrix. For the attractive case, both the bound states and the scattering states are examined. It is shown that, for n=even $\neq 0$, there exist two bound states and only one bound state for n=0. For the scattering states, it is demonstrated that one recovers the same coefficients that were obtained for the repulsive case.

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ملخص:

لقد تمّ تحليل جهود قوية القطبية في بعد واحد وبالتحديد تفاعلات المشتقة النونية $\delta^{(n)}(x)$. في حالة التنافر، تم اشتقاق معاملات الانعكاس والنفاذ. تبيّن انه، عندما n = زوجي، هذه المعاملات تحقق وحدانية مصفوفة التشتت لجميع قيم الطاقة الساقطة وان الجهد الفاصل يصبح عاكساً مثالياً لقيم محدده للطاقة الساقطه. لكن عندما n = فردي $1 \neq$ ، فان هذه المعاملات لا تحقق وحدانية مصفوفة التشتت الا عند قيمه واحده محدده للطاقة الساقطة والتي عندها يصبح الجهد الفاصل عاكساً مثالياً . أما

في حاله n = 1، فان النتائج أشارت الى ان المعاملات تحقق وحدانية مصفوفة التشتت. في حالة التجاذب، تمّ فحص كلا من حالات الربط والتشتت. تبيّن أنه في حاله n = زوجي ≠ صفر، يوجد حالتي ربط بينما يوجد حالة ربط واحده عندما n = صفر. أما حالات التشتت، فقد لوحظ انه بالامكان استرجاع المعاملات نفسها التي تم الحصول عليها في حالة التنافر.

1 -Introduction

Strongly singular potentials (or the so called zero-range) potentials or point interactions) attracted the attention of researchers since early eighties [1-6]. Quantum mechanics on graphs has been revived in the last decade [7-10], as a response to the rapid progress of fabrication techniques which allow us to produce graph-like structures of a pure semiconductor material, for which graph Hamiltonians represent a natural model. In recent years there was an emphasis on the so-called δ' interaction that is one of point interactions [11-14], and especially the scattering S-Matrix [15, 16]. Just recently, a proposed coordinate space regularization of a three-body problem with zero-range potentials was reported [17]. Furthermore, the scattering of nonrelativistic particles in three and lower dimensions was recently considered [18]. A close examination has been carried out to the boundary conditions for point interactions [11, 13, 19, 20]. Recently, the present author considered an exactly solvable model, namely the repulsive and attractive δ'' potential [21].

Recently, an intensive work has been carried out concerning gravitational radiation and black hole formation in D-dimensional space times [22-25]. This involves the D-dimensional retarded Green's function as solution of the homogeneous Einstein's equations. As is shown in [22], the D-dimensional retarded Green's function contains higher derivatives of delta functions.

The purpose of this paper is to examine a strongly singular potential, namely the n^{th} derivative of delta-function potential. In sect. 2, the repulsive $\delta^{(n)}$ potential is considered and the reflection and transmission coefficients for both n=even and n=odd are derived. In sect. 3, the attractive $\delta^{(n)}$ potential is examined and both bound states and scattering states for n=even and n=odd are considered. Sect. 4 is devoted for results and discussion.

2 -A repulsive n^{th} derivative of a delta-function potential

In this section the reflection and transmission coefficients for the n^{th} derivative of a delta-function potential are presented. The potential has the form

$$V(x) = \alpha\delta^{(n)}(x), \tag{1}$$

where α is a real positive constant and is a measure of the strength of the potential. The time-independent Schrodinger equation for this potential is

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \alpha\delta^{(n)}(x)\psi = E\psi. \tag{2}$$

Considering a particle incident from the left, the solution for equation (2) is

$$\psi(x) = \begin{cases} e^{ikx} + D\ \bar{e}^{ikx}, & x\langle 0 \\ B\ e^{ikx}, & x\rangle 0 \end{cases} \tag{3}$$

where $k^2 = 2mE/\hbar^2$, and the absence of the term $\bar{e}^{ikx}$ for $x\rangle 0$ is understood for a wave incident from the left. Griffiths [13] correctly derived the boundary conditions for $\psi(x)$ and $\psi'(x)$, namely

$$\Delta\psi = (-1)^n \frac{2m\alpha}{\hbar^2} n\overline{\psi}^{(n-1)}(0), \tag{4}$$

$$\Delta\psi' = (-1)^n \frac{2m\alpha}{\hbar^2}\overline{\psi}^{(n)}(0), \tag{5}$$

where $\overline{\psi}(0)$ is defined as

$$\overline{\psi}(0) = \frac{1}{2}\left[\psi(0^+) + \psi(0^-)\right]. \tag{6}$$

One may apply the above boundary conditions to the wave function in eq. (3). To that end, the following is derived

$$\psi^{(n-1)}(x) = (ik)^{n-1}\begin{cases} e^{ikx} + (-1)^{n-1} D\,\overline{e}^{ikx}, & x\langle 0 \\ B\ e^{ikx}, & x\rangle 0 \end{cases}$$

and thus, with the help of eq. (6),

$$\overline{\psi}^{(n-1)}(0) = \frac{1}{2}(ik)^{n-1}\left[B + 1 - (-1)^n D\right]. \tag{7}$$

$$\psi^{(n)}(0) = (ik)^n \begin{cases} 1 + (-1)^n\ D, & x\langle 0 \\ B\ , & x\rangle 0 \end{cases}$$

and thus

$$\overline{\psi}^{(n)}(0) = \frac{1}{2}(ik)^n\left[B + 1 + (-1)^n\ D\right]. \tag{8}$$

The substitution of eq. (7) into Eq. (4), and eq. (8) into eq. (5) gives

$$B-(1+D)=\frac{m\alpha}{\hbar^2}n(-ik)^{n-1}\left[B+1-(-1)^n D\right] \tag{9}$$

$$ik(B-1+D)=\frac{m\alpha}{\hbar^2}(-ik)^{n}\left[B+1+(-1)^n D\right]. \tag{10}$$

For simplicity, let

$$c=m\alpha/\hbar^2 , \tag{11}$$

and after some algebra, one can get

$$B=1+\frac{D}{1+n}\left[1-n+2cn(ik)^{n-1}\right], \tag{12}$$

$$D=\frac{2c(n+1)(-1)^n(ik)^{n-1}}{\left[2+2cn(ik)^{n-1}-(-1)^n c(ik)^{n-1}\left\{1-n+2cn(ik)^{n-1}+(-1)^n(1+n)\right\}\right]}. \tag{13}$$

In order to find the reflection coefficient $R\left(=|D|^2\right)$ and the transmission coefficient $T\left(=|B|^2\right)$ one may consider the two cases n=even and n = odd

case 1: n = even
Eq. (13) yields

$$D = \frac{c(n+1)(ik)^{n-1}}{1 + nc^2(k)^{2n-2} + c(n-1)(ik)^{n-1}} \ , \tag{14}$$

and thus the reflection and the transmission coefficients are respectively

$$R = \frac{c^2(n+1)^2 k^{2n-2}}{(1+nc^2k^{2n-2})^2 + c^2(n-1)^2 k^{2n-2}} \tag{15}$$

$$T = \frac{\left(1 - nc^2 k^{2n-2}\right)^2}{\left(1+nc^2k^{2n-2}\right)^2 + (1-n)^2 c^2 k^{2n-2}} \tag{16}$$

The special case n = 0 corresponds to the usual delta-function and our results yield the well-known values for R and T which are found in most standard quantum mechanics textbooks [22] namely

$$R = \frac{c^2}{k^2 + c^2} \quad and \quad T = \frac{k^2}{c^2 + k^2} .$$

It is interesting to note that if

$$c^2 k^{2n-2} = \frac{1}{n} \quad then \quad R = 1 \ , \tag{17}$$

and this happens when $E = \dfrac{\hbar^2}{2m}\left(\dfrac{\hbar^4}{n m^2 \alpha^2}\right)^{1/(n-1)}$, which means that the barrier becomes totally reflective. It is also very clear that R and T in eq's (15) and (16) satisfy the unitarity of the scattering matrix, i.e R+T = 1. This implies that the Hamiltonian under investigation (for n = even) is a self-adjoint operator. One could easily check that $T \to 1$ for the high-energy behavior by simply taking the limit of T as $k^2 \to \infty$.

Furthermore, eq. (15) shows that, for a given energy, the barrier becomes reflectiveless (R = 0) for very week strength parameter α (remember $c = m\alpha / \hbar^2$).

Case 2: n = odd
Eq. (13) gives for odd n

$$D = \frac{-c(n+1)(ik)^{n-1}}{1 + c^2 n k^{2n-2}} , \tag{18}$$

and, with the help of eq. (12), one can get

$$B = \frac{1 - nc^2 k^{2n-2} + c(n-1)(ik)^{n-1}}{1 + nc^2 k^{2n-2}} . \tag{19}$$

Therefore, the reflection and transmission coefficients are respectively given by

$$R = \frac{c^2 (n+1)^2 k^{2n-2}}{\left(1 + nc^2 k^{2n-2}\right)^2} , \tag{20}$$

$$T=\frac{\left(1-nc^2k^{2n-2}\right)^2+c^2(n-1)^2k^{2n-2}+2c(n-1)(-1)^{\frac{n-1}{2}}k^{n-1}\left(1-nc^2k^{2n-2}\right)}{\left(1+nc^2k^{2n-2}\right)^2}. \tag{21}$$

It is very useful to let $x=c\,k^{n-1}$, so that

$$R=\frac{(n+1)^2x^2}{\left(1+nx^2\right)^2}, \tag{22}$$

$$T=\frac{\left(1-nx^2\right)^2+(n-1)^2x^2+2(n-1)x\left(1-nx^2\right)(-1)^{\frac{n-1}{2}}}{\left(1+nx^2\right)^2}. \tag{23}$$

It is easy to show that

$$R+T=1+\frac{2x^2(n-1)^2+2x(n-1)\left(1-nx^2\right)(-1)^{\frac{n-1}{2}}}{\left(1+nx^2\right)^2}, \tag{24}$$

and thus if the second term vanishes then R and T satisfy the unitarity of the scattering matrix, i.e. R+T = 1. This occurs when

$$x(n-1)^2+(n-1)\left(1-nx^2\right)(-1)^{\frac{n-1}{2}}=0. \tag{25}$$

Clearly, n = 1 satisfies eq. (25) and in this case equations (22) and (23) give

$$R=\frac{4x^2}{\left(1+x^2\right)^2} \quad , \quad T=\frac{\left(1-x^2\right)^2}{\left(1+x^2\right)^2} \quad , \tag{26}$$

which means that the Hamiltonian for the $\delta'(x)$ potential is a self-adjoint operator for all values of x.
For $n \neq 1$, eq. (25) is rewritten as

$$nx^2-(n-1)(-1)^{\frac{n-1}{2}}x-1=0 \quad , \tag{27}$$

whose positive root is

$$x=\frac{n\left(1+(-1)^{\frac{n-1}{2}}\right)+\left(1-(-1)^{\frac{n-1}{2}}\right)}{2n} \quad , \tag{28}$$

and the negative root is neglected since x is positive only. Eq. (28) shows two cases:

a) n = 3 Mod 4, then x = 1/n and hence eq's (22) and (23) yield R = 1 and T = 0. This implies that in this case (n = 3 mod 4) the barrier is perfectly reflective when $c\,k^{n-1}=1/n$ and hence the corresponding Hamiltonian is a self-adjoint operator. This occurs if and only if $E=\frac{\hbar^2}{2m}\left(\hbar^2/nm\alpha\right)^{2/(n-1)}$.

b) n = 5 Mod 4, then x = 1, and hence eq's (22) and (23) give R = 1 and T = 0. Therefore. In this case, the barrier is perfectly reflective when $c\,k^{n-1}=1$ and thus the corresponding Hamiltonian is a self-adjoint operator. This occurs when $E = \frac{\hbar^2}{2m}\left(\frac{\hbar^2}{m\alpha}\right)^{2/(n-1)}$. A final remark for the n=odd case is that one could easily check that $R \to 0$ and $T \to 1$ for the high energy behavior by simply taking the limits of R and T as $x \to \infty$ in equations (22) and (23. Also, for a given energy the barrier becomes reflectiveless $(R \to 0)$ for very weak strength parameter α .

3 - An attractive n^th derivative of a delta-function potential

Consider a potential of the form

$$V(x) = -\alpha\delta^{(n)}(x) \tag{29}$$

The Schrodinger equation reads

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} - \alpha\delta^{(n)}(x)\psi(x) = E\psi(x), \tag{30}$$

and one may need to examine both the bound states $(E\langle 0)$ and the scattering states $(E\rangle 0)$.

3.1. Bound states: Equation (30), with $k^2 = 2m|E|/\hbar^2$ and $c \equiv m\alpha/\hbar^2$ as before, can be written as:

$$\frac{d^2\psi(x)}{dx^2} - k^2\psi(x) = -2c\,\delta^{(n)}(x)\psi(x). \tag{31}$$

For $x \neq 0$, we have

$$\frac{d^2\psi(x)}{dx^2} - k^2\psi(x) = 0, \tag{32}$$

whose general solution is

$$\psi(x) = \begin{cases} A\ \overline{e}^{kx} & x\rangle 0 \\ B\ e^{kx}\ , & x\langle 0 \end{cases} \tag{33}$$

Here it is assumed that $\psi(x)$ is squared integrable and hence vanishes as $x \to \pm\infty$. In order to apply the boundary conditions given in equations (4) and (5), one calculates

$$\psi^{(n)}(x) = \begin{cases} A(-k)^n\ \overline{e}^{kx}\ , & x\rangle 0 \\ B(k)^n\ e^{kx}\ , & x\langle 0 \end{cases}$$

$$\psi^{(n-1)}(x) = \begin{cases} A(-k)^{n-1}\ \overline{e}^{kx}\ , & x\rangle 0 \\ B(k)^{n-1}\ e^{kx}\ , & x\langle 0 \end{cases}$$

and thus, with the help of eq. (6), one can find

$$\overline{\psi}^{(n)}(0) = \frac{1}{2}\left[A(-k)^n + B(k)^n\right]$$

$$\overline{\psi}^{(n-1)}(0) = \frac{1}{2}\left[A(-k)^{n-1} + B(k)^{n-1}\right].$$

Therefore, the substitution of the above relations into equations (4) and (5) yields

$$A - B = n\,c(-k)^{n-1}\left[A(-1)^{n-1} + B\right], \tag{34}$$

$$-k(A+B) = (-1)^n\, c\, k^n\left[A(-1)^n + B\right]. \tag{35}$$

Solving the above equations gives

$$c - 2(-k)^{1-n} = c\left[2n\,c(k)^{n-1} - (n-1)(-1)^{n-1} + n\right], \tag{36}$$

which yields the energy of the bound states. One may consider both cases n = even and n = odd.

Case 1: n = even

First the special case n = 0 is considered separately. Eq. (36) for n = 0 gives k = -c and thus by substituting the values of k and c, one can get,

$$E = -\frac{m\alpha^2}{\hbar^2}\ . \tag{37}$$

So the attractive delta function potential has one bound state with the above energy as is well-known in standard quantum mechanics textbooks [22]. Eq. (36) for n = even can be written as

$$n\,x^2 + (n-1)x - 1 = 0\ , \tag{38}$$

where as before $x = c\,k^{n-1}$. The roots of the above equation are x = -1 and 1/n. For $x = -1 : k^{n-1} = -1/c$ and this gives

$$E = \frac{-\hbar^4}{m^2\alpha^2}\left(\frac{\hbar^2}{2m}\right)^{\frac{1}{n-1}} . \tag{39}$$

For $x = \frac{1}{n}$: $k^{n-1} = \frac{1}{n\,c}$ and this gives

$$E = \frac{-\hbar^4}{n^2 m^2\alpha^2}\left(\frac{\hbar^2}{2m}\right)^{\frac{1}{n-1}} . \tag{40}$$

This indicates that there are two bound states for each even n with the above energies.

Case 2: n = odd
Eq. (36) yields

$$k^{2(n-1)} = \frac{-1}{n\,c^2} \ , \tag{41}$$

and since k^2 is negative and n = odd, the quantity $k^{2(n-1)}$ is positive. Therefore eq. (41) can not hold for real c. This implies that the potential for n = odd has no bound states.

3.2. Scattering states.

The simplest way to study scattering states is to change the sign of α in the repulsive case discussed in section 2. This implies that c must be replaced by –c. By so doing, the reflection and transmission coefficients, given in equations (15) and (16), for n = even remain unchanged. The case n = odd needs little modification but the outcome physics remains the same. First the special case n = 1 is unchanged, as it seen from eq. (26), when x is replaced by –x. For $n \neq 1$, the reflection coefficient remains unchanged, see eq. (22), but the transmission coefficient is, slightly changed. The third term in eq. (23) would acquire a negative sign when x is replaced by –x. Therefore the requirement for the unitarity of the scattering matrix, given in eq. (27), becomes

$$n x^2 + (n-1)(-1)^{\frac{n-1}{2}} x - 1 = 0 \ , \tag{42}$$

whose positive root is

$$x = \frac{n\left(1-(-1)^{\frac{n-1}{2}}\right)+\left(1+(-1)^{\frac{n-1}{2}}\right)}{2n} \ . \tag{43}$$

Eq. (43) now shows that if n = 3 Mod 4 then x = 1 and hence eq's (22) and (23) yield R = 1 and T = 0. This means that the barrier is perfectly reflective when $x\left(= c\,k^{n-1}\right) = 1$ and thus the corresponding Hamiltonian is self-adjoint. The above root also shows that if n = 5 Mod 4 then x = 1/n, and hence eq's (22) and (23) yield R=1 and T=0. Again, the barrier in this case is perfectly reflective when $c\,k^{n-1} = 1/n$, and the corresponding Hamiltonian is self-adjoint. Therefore for the scattering states the replacement of α by -α keeps the physics outcome unchanged. This shows that the particle is just as likely to pass through the barrier as to cross over the well when $c\,k^{n-1}$ is chosen properly. Thus, all our remarks concerning the high energy behavior and the weak strength parameter remain the same as for the repulsive potential case.

4 - Results and discussion

In this paper an exactly solvable model of point interactions in one dimension is considered. Namely the n^{th} derivative of a delta function potential is analyzed. For the repulsive case, the transmission and reflection coefficients are derived. For n=even: It was shown that R and T satisfy the unitarity of the scattering matrix for all values of the incident energy which implies that the corresponding Hamiltonian is self-adjoint. Our results yield the well-known values of R and T of the repulsive delta function potential, i.e when n=0. The results show that the barrier becomes perfectly reflective when the incident particle has energy $E=\left(\frac{\hbar^4}{nm^2\alpha^2}\right)^{1/(n-1)}$.

For n=odd: Our results for n=1 (i.e δ' potential) yield reflection and transmission coefficient that satisfy the unitarity of the scattering matrix for all values of the incident particle's energy. For n=3 Mod 4, the barrier is perfectly reflective if and only if the incident particle has energy $E=\frac{\hbar^2}{2m}\left(\frac{\hbar^2}{nm\alpha}\right)^{2/(n-1)}$. For n=5 Mod 4, the barrier is perfectly reflective if and only if the incident particle has energy $E=\frac{\hbar^2}{2m}\left(\frac{\hbar^2}{m\alpha}\right)^{2/(n-1)}$. Thus for n=odd $\neq 1$, the corresponding Hamiltonian is not self-adjoint except for the above two values of the incident energy. It was also shown that for both cases of n even and odd the barrier becomes reflectiveless for the high energy behavior and for the very weak strength parameter α.

For the attractive potential, the bound states for n=even were derived and it was found that there is only one bound state when n=o and two bound states when $n \neq 0$. For n=odd it was shown that there is no bound states and thus the potential can not bind the particle. The scattering states were considered and it was shown that the particle is just likely to pass through the barrier as to cross over the well.

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